

# Stability of local quantum dissipative systems

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## Abstract

Open quantum systems weakly coupled to the environment are modelled by completely positive, trace preserving semigroups of linear maps. The generators of such evolutions are called Lindbladians. For practical and theoretical reasons, it is crucial to estimate the impact that noise or errors in the generating Lindbladian can have on the evolution. In the setting of quantum many-body systems on a lattice it is natural to consider local or exponentially decaying interactions. We show that in this case local observables and correlation functions are stable if the Lindbladian is frustration free, translational invariant (uniformly), has a unique fix point (with no restriction on its rank) and has a mixing time which scales logarithmically with the system size. These conditions can be relaxed to the non-translational invariant case, at the cost of requiring *Local Topological Quantum Order*. As a main example we prove that classical Glauber dynamics is stable under local perturbations, including perturbations in the transition rates which do not preserve detailed balance.

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# 1 Background and previous work

The physical properties of a closed many-body quantum system are encoded in its Hamiltonian. Theoretical models of such systems typically assume some form of local structure, whereby the Hamiltonian decomposes into a sum over interactions between subsets of nearby particles. Similarly, the behaviour of an open many-body quantum system is encoded in its Liouvillian. Again, this is typically assumed to have a local structure, decomposing into a sum over local Liouvillians acting on subsets of nearby particles.

Crucial to justifying such theoretical models is the question of whether their physical properties are stable under small perturbations to the local interactions. If the physical properties of a many-body Hamiltonian or Liouvillian depend sensitively on the precise mathematical form of those local terms, then it is difficult to conclude anything about real physical systems, whose interactions will always deviate somewhat from theory.

Quantum information theory has motivated another perspective on many-body Hamiltonians. Rather than studying models of naturally occurring systems, it studies how many-body systems could be engineered to produce desirable behaviour, such as long-term storage of information in quantum memories [8, 15, 16, 38], processing of quantum information for quantum computing [7, 9, 10, 29, 41], or simulation of other quantum systems which are computationally intractable by classical means [2, 4, 5, 22, 24]. Again, stability of these systems under local perturbations is crucial, otherwise even tiny imperfections will destroy the desired properties. This has been studied particularly in the context of self-correcting topological quantum memories, where one requires stability

also against local sources of *dissipative* noise, and the relevant quantity is the time it takes to introduce logical errors in the system. It has been known since [1, 8] that a self-correcting quantum memory with local interactions is possible in four spatial dimensions. With the breakthrough given by the Haah code [15], it now seems plausible to find such a self-correcting quantum memory in 3D.

Recently, and partly motivated by the dissipative nature of noise, this approach has been extended to open quantum systems and many-body Liouvillians. First theoretical [30, 47] and then experimental [3, 31] work showed that creating many-body quantum states as fixed points of engineered dissipative Markovian evolutions could be more robust against undesirable errors and maintain coherence for longer. Intuitively, there is an inherent robustness in such models: the target state is independent of the initial state. If the dissipation is engineered perfectly, the system will always be driven back towards the desired state. This idea can be used to engineer dissipative systems both for storing quantum information and for carrying out computation via dissipative dynamics. However, it does not guarantee stability against errors *in the engineered Liouvillian itself*. Once again, stability against local perturbations – this time for many-body Liouvillians rather than Hamiltonians – is of crucial importance.

In the case of closed systems governed by Hamiltonians, major recent breakthroughs have given rigorous mathematical justification for believing in stability of the physical properties of many-body Hamiltonians. Starting with [6], it culminated in the work of [39] which proved that, under a set of mathematically well-defined and physically reasonable conditions, gapped many-body Hamiltonians are stable under perturbations to the local interactions.<sup>1</sup> More precisely, in the presence of *frustration-freeness*, *local topological quantum order*, and *local gap*, the spectral gap of a Hamiltonian with (quasi) local interactions is stable against small (quasi) local perturbations (see Section 4 for a formal definition of these conditions). The bound on the amount of imperfection tolerated by the system comes in terms of the local gap, the decay of the local topological order, and the strength (and decay rate) of the interactions. Furthermore, except for frustration freeness, which is a technical condition required in the proof, these conditions are in a sense tight. There exist simple counterexamples to stability if any one of the conditions is lifted.

## 2 Stability of open quantum systems

In this work, we study stability of many-body Liouvillians. Our main result shows that under analogous assumptions to those in [39], one can prove stability also in dissipative systems.

However, although the result is analogous, the proof and even the definition of stability in the case of Liouvillians necessarily differ substantially from the Hamiltonian case. For Hamiltonians, the relevant issue is stability of the spectral gap. Via the quasi-adiabatic technique [18, 20], this in turn implies a smooth transition between the initial and perturbed ground states, showing that both are within the same phase. Note that the existence of a smooth transition does

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<sup>1</sup>Note that, in stark contrast to traditional perturbation theory, the perturbations considered here simultaneously change *all* the local interactions by a small amount. The perturbations are therefore unbounded, and standard perturbation theory is useless. It is the local structure of the Hamiltonian and the perturbations that ensures stability.

not imply that both ground states are close in norm, as the trivial example  $H = \otimes |0\rangle\langle 0|$  vs.  $H = \otimes(|0\rangle + \varepsilon |1\rangle)(\langle 0| + \varepsilon \langle 1|)$  shows. It does however imply a well-behaved perturbation in the expectation of local observables – such as order parameters – and correlation functions, which in most experimental situations are the only measurable quantities.

For Liouvillians, we are interested in a definition of stability more related to the evolution itself, which accounts at the same time for both the speed of convergence and the properties of the fixed point. Here, we consider the strongest definition of stability: we want our systems (initial and perturbed) to evolve similarly for all times and all possible initial states. Thus, not only should the speed of convergence to the fixed points be similar, the fixed points themselves should be close and so should the approach to the fixed points.

This definition is significantly stronger than stability of the spectral gap alone<sup>2</sup>, and is more directly relevant to the applications discussed above. As in the Hamiltonian case, the analogous trivial example shows that one cannot expect to attain such stability if we consider global measurements on the whole system. We therefore restrict our attention to local observables and few-body correlation functions.

Furthermore, there are subtleties involved in extending this stronger definition of stability to dynamics with multiple fixed points. Rather than getting side-tracked by these, we defer consideration of multiple fixed points to a future paper, and restrict our attention to dissipative dynamics with unique fixed points. It is important to note, however, that we do not make *any* assumption on the form of the unique fixed point. In particular, we do not assume that it is full-rank (primitivity); our results apply equally well to Liouvillians with pure fixed points. (Pure-state fixed points are particularly relevant to quantum information applications, such as dissipative state engineering and computation.)

A key technical ingredient in the stability proof for Hamiltonians is the quasi-adiabatic evolution technique [18, 20], which directly uses the fact that Hamiltonian dynamics is reversible. This is of course no longer true for Liouvillians, so we must use a different proof approach. We make use of the fact that evolution under a Liouvillian converges to a steady-state, together with dissipative generalizations [40] of the Lieb-Robinson bounds that are the other crucial ingredient in [39].

Among the systems which verify our hypotheses, one finds classical Glauber dynamics [37]. This immediately shows that Glauber dynamics is stable against errors. To the best of our knowledge, this is new even to the classical literature. Given the importance of Glauber dynamics to sampling from the thermal distributions of classical spin systems [33, 37], we expect our results to have applications also to classical statistical mechanics.

The paper is structured as follows. After setting up notation and basic definitions in the next section, we state our main stability result in section 4 and discuss the assumptions it requires. In section 5 we prove various technical results used in the main proof, which is given in section 6. We apply these results in section 7 to the important example of classical Glauber dynamics, before concluding with a discussion of the results and related open questions in section 8.

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<sup>2</sup>Also note that by the recent work [44] it is not clear whether the spectral gap in Liouvillians is the relevant quantity for convergence questions.

### 3 Setup and notation

We will consider a quantum system defined on a square lattice<sup>3</sup>  $\Lambda = [0, L-1]^D \subset \mathbb{Z}^D$  or  $\Lambda = (\mathbb{Z}/L\mathbb{Z})^D \subset \mathbb{T}^D$ . The ball centered at  $x \in \Lambda$  of radius  $r$  will be denoted by  $b_x(r)$ . We will denote by  $N$  the number of sites in the lattice, i.e.  $N = L^D$ . At each site  $x$  of the lattice we will associate one elementary quantum system with Hilbert space  $\mathcal{H}_x$ . Then for each subset  $\Gamma \subseteq \Lambda$  the associated Hilbert space will be

$$\mathcal{H}_\Gamma = \bigotimes_{x \in \Gamma} \mathcal{H}_x,$$

and the algebra of observables supported on  $\Gamma$  is defined by

$$\mathcal{A}_\Gamma = \bigotimes_{x \in \Gamma} \mathcal{B}(\mathcal{H}_x).$$

If  $\Gamma_1 \subset \Gamma_2$ , there is a natural inclusion of  $\mathcal{A}_{\Gamma_1}$  in  $\mathcal{A}_{\Gamma_2}$  by identifying it with  $\mathcal{A}_{\Gamma_1} \otimes \mathbb{1}$ . The support of an observable  $O \in \mathcal{A}_\Lambda$  is the minimal set  $\Gamma$  such that  $O = O' \otimes \mathbb{1}$ , for some  $O' \in \mathcal{A}_\Gamma$ , and will be denoted by  $\text{supp } O$ . We will denote by  $\|\cdot\|_p$  the Schatten  $p$ -norm over  $\mathcal{A}_\Lambda$ . Where there is no risk of ambiguity,  $\|\cdot\|$  will denote the usual operator norm (i.e. the Schatten  $\infty$ -norm).

A linear map  $\mathcal{T} : \mathcal{A}_\Lambda \rightarrow \mathcal{A}_\Lambda$  will be called a *superoperator*. The support of a superoperator  $\mathcal{T}$  is the minimal set  $\Gamma \subseteq \Lambda$  such that  $\mathcal{T} = \mathcal{T}' \otimes \mathbb{1}$ , where  $\mathcal{T}' \in \mathcal{B}(\mathcal{A}_\Gamma)$ . A superoperator is said to be Hermiticity preserving if it maps Hermitian operators to Hermitian operators. It is said to be positive if it maps positive operators (i.e. operators of the form  $O^*O$ ) to positive operators.  $\mathcal{T}$  is called *completely positive* if  $\mathcal{T} \otimes \mathbb{1} : \mathcal{A}_\Lambda \otimes M_n \rightarrow \mathcal{A}_\Lambda \otimes M_n$  is positive for all  $n \geq 1$ . Finally, we say that  $\mathcal{T}$  is trace preserving if  $\text{tr } \mathcal{T}(\rho) = \text{tr } \rho$  for all  $\rho \in \mathcal{A}_\Lambda$ . For a general review on superoperators, see [49].

The dynamics of the system is generated by a superoperator  $\mathcal{L}$ , which plays a similar role to the Hamiltonian in the non-dissipative case. The evolution will be given by the one parameter semigroup  $T_t = e^{t\mathcal{L}}$ . The natural assumptions to make about  $T_t$  are that it is a continuous semigroup of completely positive and trace preserving maps (CPTP, sometimes also called *quantum channels*). Such maps are always contractive, meaning that  $\|T_t\|_{1 \rightarrow 1, cb} \leq 1$ , where:

$$\|T\|_{1 \rightarrow 1, cb} = \sup_n \|T \otimes \mathbb{1}_n\|_{1 \rightarrow 1} = \sup_n \sup_{\substack{X \in \mathcal{A}_\Lambda \otimes M_n \\ X \neq 0}} \frac{\|T \otimes \mathbb{1}_n(X)\|_1}{\|X\|_1}. \quad (1)$$

We will also be interested in the  $\|\cdot\|_{\infty \rightarrow \infty, cb}$  norm of superoperators, which is defined as follows:

$$\|T\|_{\infty \rightarrow \infty, cb} = \sup_n \|T \otimes \mathbb{1}_n\|_{\infty \rightarrow \infty} = \sup_n \sup_{\substack{X \in \mathcal{A}_\Lambda \otimes M_n \\ X \neq 0}} \frac{\|T \otimes \mathbb{1}_n(X)\|_\infty}{\|X\|_\infty}.$$

The relationship between  $\|\cdot\|_{1 \rightarrow 1, cb}$  and  $\|\cdot\|_{\infty \rightarrow \infty, cb}$  is the following

$$\|T\|_{1 \rightarrow 1, cb} = \|T^*\|_{\infty \rightarrow \infty, cb}.$$

We will denote  $\|\cdot\|_{\infty \rightarrow \infty, cb}$  simply by  $\|\cdot\|_{cb}$  when there is no risk of confusing different completely-bounded norms.

<sup>3</sup>We restrict to square lattices for the sake of exposition. The results can be extended to more general settings, replacing the lattice  $[0, L-1]^D$  with a graph equipped with a sufficiently regular metric.

*Observation 3.1.* As shown in [23], the supremum in equation (1) is reached when  $n$  is equal to the dimension of the space on which  $T$  is acting: if  $T : \mathcal{M}_n \rightarrow \mathcal{M}_n$ , then  $\|T \otimes \mathbb{1}_n\|_{1 \rightarrow 1} = \|T\|_{1 \rightarrow 1, cb}$ .

The generator  $\mathcal{L}$  of such a semigroup is called a *Lindbladian*. All such generators can be written in the following general form, often called the *Lindblad form* [11, 34] (see [49]):

**Proposition 3.2.**  $\mathcal{L}$  generates a continuous semigroup of CPTP maps if and only if it can be written in the form:

$$\mathcal{L}(\rho) = i[\rho, H] + \sum_j L_j \rho L_j^* - \frac{1}{2} \{L_j^* L_j, \rho\}, \quad (2)$$

where  $H$  is an Hermitian matrix,  $\{L_j\}_j$  a set of matrices called the *Lindblad operators*,  $[\cdot, \cdot]$  denotes the commutator and  $\{\cdot, \cdot\}$  the anticommutator.

Since  $\mathcal{L}$  is defined on a lattice  $\Lambda$ , it is natural to ask that it has some form of local structure. We will say that  $\mathcal{L}$  is a *local Lindbladian* if it can be written as a sum of terms, each of which is itself in Lindblad form and has bounded support:

$$\mathcal{L} = \sum_u \sum_r \mathcal{L}_u(r), \quad \text{supp } \mathcal{L}_u(r) = b_u(r). \quad (3)$$

Such a decomposition is obviously always trivially possible. We are interested in the cases in which the norms of  $\mathcal{L}_u(r)$  decay with  $r$ . Concretely, let us define the *strength of interaction* as the pair  $(J, f(r))$  given by

$$J = \sup_{u, r} \|\mathcal{L}_u(r)\|_{1 \rightarrow 1, cb}, \quad f(r) = \sup_u \frac{\|\mathcal{L}_u(r)\|_{1 \rightarrow 1, cb}}{J}. \quad (4)$$

We will restrict our attention to the case in which  $f(r)$  is compactly supported, which is usually called the *finite range interaction* case

$$f(r) = 0 \quad \forall r > k, \quad \text{for some finite } k,$$

or the case in which  $f(r)$  is decaying exponentially fast,

$$f(r) = e^{-\mu r} \quad \text{for some } \mu > 0.$$

More generally, we will say that a function  $f(r)$  is *fast-decaying* if it goes to zero faster than any polynomial<sup>4</sup>, as  $r$  goes to infinity. In such case we say that  $\mathcal{L}$  has *quasi-local interactions*. Quasi-local interactions verify a Lieb-Robinson bound (see section 5.1) with slower decay in space than the previous classes of interactions. For the sake of simplicity of the exposition, we will not go into such generalizations. Nonetheless, the proof presented in this paper applies, with the obvious modifications, to this setting.

As shown in [48], from the spectral decomposition of  $\mathcal{L}$  (and  $T_t$ ) one can define two new CPTP maps which represent the infinite-time limit of the semigroup  $T_t$ . We will denote by  $T_\infty$  the projector on the subspace of stationary states (fixed points), and by  $T_\phi$  the projector on the subspace of periodic states. They

<sup>4</sup>Or more generally, if it decays faster than a polynomial of some degree, which depends on the geometry of the lattice.

correspond, respectively, to the kernel of  $\mathcal{L}$  and to the eigenspace of purely imaginary eigenvalues of  $\mathcal{L}$ , which we denote  $\mathcal{F}_{\mathcal{L}}$  and  $\mathcal{X}_{\mathcal{L}}$ , respectively. Both the subspaces are invariant under  $T_t$ : in particular,  $T_t$  acts as the identity over  $\mathcal{F}_{\mathcal{L}}$ , while it is a unitary operator over  $\mathcal{X}_{\mathcal{L}}$ . Note that both subspaces are spanned by positive operators (i.e. density matrices).

As we are going to exploit the local structure of  $\mathcal{L}$ , we will often be interested in the restriction of  $\mathcal{L}$  to a subset of the lattice. Given  $A \subset \Lambda$ , we define the *truncated*, or *localized*, generator:

$$\mathcal{L}_A = \sum_{b_u(r) \subseteq A} \mathcal{L}_u(r). \quad (5)$$

We will use superscripts to indicate truncated versions of the CPTP maps generated by  $\mathcal{L}_A$ , so that  $T_t^A$ ,  $T_\infty^A$  and  $T_\phi^A$  indicate respectively the semigroup generated by the truncated generator, the projector on its fixed point subspace, and the projector on the periodic subspace of  $e^{t\mathcal{L}_A}$ .

**Definition 3.3.** *As usual, we define the spectral gap of  $T_t$  to be the difference between 1, and the maximum modulus of the elements of the spectrum of  $T_t$  strictly contained in the unit disk (i.e. the spectrum of  $T_t$  minus its peripheral spectrum). We say that  $T_t$  is gapped if its spectral gap does not vanish with increasing system size.*

**Definition 3.4.** *We say that  $\mathcal{L}$  has a unique fixed point if  $\mathcal{X}_{\mathcal{L}} = \mathcal{F}_{\mathcal{L}} = \{\rho_\infty\}$ . In other words, the spectrum of  $\mathcal{L}$  has only one element with modulus 1; namely, the non-degenerate eigenvalue 1.*

## 4 Main result

### 4.1 Assumptions for stability

In order to prove stability of the generated dynamics, we will impose three conditions on the local Lindbladian. (These assumptions are analogues of the conditions for the stability of Hamiltonians imposed in [39], and in some cases are direct generalizations of these to the dissipative setting.)

**Definition 4.1** (Frustration-Freeness). *We say that  $\mathcal{L}$  is frustration free if for all  $A \subset B \subset \Lambda$ , we have the following inclusion property for the periodic subspaces of the local Lindbladians:*

$$\mathcal{X}_{\mathcal{L}_B} \subset \mathcal{X}_{\mathcal{L}_A}. \quad (6)$$

Topological quantum order, namely the property of some quantum states to be indistinguishable on a small scale, is a widely studied property of ground states of local Hamiltonians. In the dissipative setting there are no longer any ground states, but we can easily define the analogous concept for periodic states of Lindbladians.

**Definition 4.2** (Local Topological Quantum Order (LTQO)).

*Take a convex set  $A \subset \Lambda$  and let  $A(\ell) = \{x \in \Lambda \mid \text{dist}(x, A) \leq \ell\}$ . Given two states  $\rho_i \in \mathcal{X}_{\mathcal{L}_{A(\ell)}}$ ,  $i = 1, 2$ , consider their reduced density matrices on  $A$ :*

$$\rho_i^A = \text{tr}_{A(\ell) \setminus A} \rho_i, \quad i = 1, 2.$$

We say that  $\mathcal{L}$  has local topological quantum order (LTQO) if for each  $\ell \geq 0$ :

$$\|\rho_1^A - \rho_2^A\|_1 \leq k(|A|)\Delta_0(\ell), \quad (7)$$

where  $\Delta_0(\ell)$  is a fast-decaying function and  $k(|A|)$  is polynomial in  $|A|$ .

In Hamiltonian systems, the spectral gap (the difference between the two lowest energy levels) plays a crucial role in a number of settings, from defining quantum phases and phase transitions [43] to understanding the entanglement and correlations present in the system [17, 19, 21] or analyzing its stability [6, 39]. On the other hand, it is known that, for Lindbladians, the spectral gap alone is not sufficient to fully characterize the convergence properties of the dissipative evolution [27, 44]. Therefore, we will instead impose a more general requirement on the convergence of the dynamics. (It is not known – though it seems unlikely – whether this requirement depends only on simple spectral properties of  $\mathcal{L}$ , i.e. properties depending on the eigenvalues – like the gap – and eigenvectors – like the condition number.)

**Definition 4.3** (Truncated rapid mixing). *Define the contraction of  $T_t$ ,*

$$\eta(T_t) = \sup_{\substack{\rho \geq 0 \\ \text{tr } \rho = 1}} d_{\text{tr}}(T_t(\rho), T_t T_\phi(\rho)), \quad (8)$$

where  $d_{\text{tr}}$  denotes the trace distance  $d_{\text{tr}}(\rho, \sigma) = \frac{1}{2} \|\rho - \sigma\|_1$ . We say that a Lindbladian  $\mathcal{L}$  verifies truncated rapid mixing if for any  $x \in \Lambda$  and any  $r \geq 0$  we have that

$$\eta(T_t^{b_r(x)}) \leq k(r)e^{-t\gamma}, \quad (9)$$

for some  $\gamma > 0$  and some  $k(r) > 0$  polynomial in  $r$ .

Note that this definition implies that the global mixing time, i.e. the quantity  $t_\varepsilon = \min\{t > 0 \mid \eta(T_t) < \varepsilon\}$ , scales logarithmically with the system size. As we will see later, this seems to leave out interesting systems, for example the ones associated with dissipative state engineering of topological codes [47]. On the other hand a number of systems do verify this kind of condition, and new techniques are being developed that provide tools to show such convergence rates, like the use of quantum logarithmic Sobolev inequalities and hypercontractivity [25].

Up to now, we have made no assumption on how the local terms  $\mathcal{L}_u(r)$  at different lattice sites relate to each other, other than the very generic condition on the decay of their norms in  $r$ . Nor have we assumed anything about how  $\mathcal{L}$  is defined for different system sizes. This very generic setting implies that equation (9) must be verified for all subsystems  $b_u(r)$ . For complex models this could be prohibitive. On the other hand, realistic models usually do not need such a generic formulation: the dynamics on different subsystems are typically related, or even identical in the most common case of full translational invariance. In section 6.2 we will give a precise definition of families of translationally-invariant local Lindbladians with arbitrary boundary conditions, which we call *uniform*, such that the truncated dynamics  $T_t^{b_u(r)}$  is similar to the global dynamics  $T_t$  on a smaller lattice. For these Lindbladians, which in particular includes all translationally-invariant models with arbitrary boundary conditions, the assumption of definition 4.3 can be weakened.



**Definition 4.4** (Global rapid mixing). *We say that a Lindbladian  $\mathcal{L}$  verifies global rapid mixing if*

$$\eta(T_t) \leq k(|\Lambda|)e^{-t\gamma}; \quad (10)$$

*for some  $\gamma > 0$  and some  $k(r) > 0$  polynomial in  $r$ .*

This last assumption can be restated as a logarithmic scaling with system size of the mixing time  $t_\varepsilon$  for  $T_t$ . Remarkably, if we assume that  $\mathcal{L}$  has a unique fixed point for each system size (an assumption that will be required in the proof of the stability theorem), then global rapid mixing is already sufficient to prove LTQO, for arbitrary boundary conditions. This is one of the first results we are aware of which gives conditions to prove LTQO.

**Theorem 4.5.** *Let  $\mathcal{L}$  belong to a uniform family of local Lindbladians verifying global rapid mixing, frustration freeness, and having a unique stationary state. Then  $\mathcal{L}$  verifies LTQO.*

The proof of this result, together with detailed definitions, will be given in sections 6.2 and 6.3.

## 4.2 Stability

With the required assumptions laid out, we can now state our main result.

**Theorem 4.6.** *Let  $\mathcal{L}$  be a local Lindbladian with a unique fixed point, verifying frustration-freeness and either of the following conditions:*

- (a) *LTQO and truncated rapid mixing; or*
- (b)  *$\mathcal{L}$  belongs to a uniform family verifying global rapid mixing.*

*Let  $E = \sum_u \sum_r E_u(r)$  be a sum of super-operators, which we will call the perturbation. Suppose the perturbation verifies the following assumptions:*

- (i)  *$\|E_u^*(r)\|_{cb} \leq \mathcal{E}f(r)$ , where  $\mathcal{E}$  is a constant (the “strength” of the perturbation) and  $f(r)$  is a fast-decaying function.*
- (ii)  *$E_u^*(r)[\mathbb{1}] = 0$ .*
- (iii)  *$e^{t(\mathcal{L}+E)}$  is a contraction for each  $t \geq 0$ .*

*For an observable  $O_A$  supported on  $A \subset \Lambda$ , let*

$$O_0(t) = e^{t\mathcal{L}^*}(O_A), \quad \text{and} \quad O_1(t) = e^{t(\mathcal{L}^*+E^*)}(O_A).$$

*Then we have that for all  $t \geq 0$*

$$\|O_0(t) - O_1(t)\| \leq c(|A|) \|O_A\| \mathcal{E}, \quad (11)$$

*for some  $c$  not depending on the system size, independent of  $t$ , and polynomial in  $|A|$ .*

**Observation 4.7.** The assumptions (ii)-(iii) on the perturbation  $E$  are satisfied if each  $\mathcal{L}_u(r) + E_u(r)$  is a Lindbladian, but there are more general perturbations which are covered by the theorem.

*Observation 4.8.* Since we are free to choose an  $O_A$  with support on two non connected regions, we can apply theorem 6.11 to two-point correlation functions (or more generally  $k$ -point correlation functions, for fixed  $k$ ) and still obtain that the error introduced by the perturbation depends linearly on the strength of the perturbation (and not on its global norm).

One of the tools already known in the setting of classical Markov chain [12–14, 37], and recently generalized to quantum dissipative systems [27], are the so called Logarithmic Sobolev inequalities (in short, Log Sobolev inequalities). Introduced in a different setting to study hypercontractivity of semigroups [28], they provide the right asymptotic regime needed to verify the global rapid mixing hypothesis: in fact, the existence of a system size independent Log Sobolev constant implies a logarithmic scaling of the mixing time, which is exactly what is required in Definition 4.4. Without going in the technical details of Log Sobolev inequalities, we summarize this fact in the following Corollary.

**Corollary 4.9.** *Let  $\mathcal{L}$  belong to a uniform family of Lindbladians, verifying frustration-freeness and having a unique fixed point for each system size. If  $\mathcal{L}$  verifies the Log Sobolev inequality with a system-size independent constant, then the system is stable, in the sense of theorem 4.6.*

A straightforward consequence of the techniques we use to prove stability, is that the fixed points local Lindbladians which satisfy the assumptions of theorem 4.6 necessarily have fast decay of correlations.<sup>5</sup>

**Theorem 4.10.** *Let  $\mathcal{L}$  be a local Lindbladian satisfying either of the sets of conditions of theorem 4.6. Let  $O_A$  and  $O_B$  be operators supported on regions  $A$  and  $B$ , with  $\|O_A\| = \|O_B\| = 1$ , and call  $d_{AB}$  be the distance between them. Define the correlation function*

$$\text{corr}(O_A, O_B) = |\langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle|,$$

where  $\langle O \rangle = \text{tr}(\rho_\infty O)$  is the expectation value of the operator  $O$  acting on the fixed point  $\rho_\infty$  of  $\mathcal{L}$ . Then we have that

$$\text{corr}(O_A, O_B) \leq p(|A| + |B|)g(d_{AB}),$$

where  $g(d)$  is a fast-decaying function and  $p(\cdot)$  is growing polynomially. Moreover, if  $\mathcal{L}$  verifies global rapid mixing, or verifies LTQO with an exponentially decaying  $\Delta_0$ , then  $g(d)$  is also exponentially decaying.

### 4.3 Local observables vs. global observables

The bound in equation (11) has a scaling on the size of the support of the observable  $B$ . Although the dependence is polynomial, for observables with large support the result is not useful. In most realistic experiments, only finite support observables and  $k$ -site correlation functions can be measured, so this is not such an issue in practice. Nonetheless, one might ask more generally for a bound on  $\sup_\rho d_{\text{tr}}(T_\infty(\rho), T'_\infty(\rho))$ , where  $T'_\infty$  is the fixed-point projector for the perturbed  $\mathcal{L}' = \mathcal{L} + E$ .

---

<sup>5</sup>As we were writing up this manuscript, a result similar to Theorem 4.10 was announced by Kastorayno et al. [26].

However, this is not possible; the limitation to local observables is in some sense strict. There is no hope of finding such a bound for global observables, as the following simple example shows.<sup>6</sup>

*Example 4.11.* Consider  $N$  independent amplitude damping processes, with uniform rate  $\gamma$  (which we can suppose w.l.o.g. equal to 1). This Lindbladian can be written as

$$\mathcal{L}_N = \sum_{k=1}^N \mathbb{1}_{1\dots k-1} \otimes \mathcal{L}_1 \otimes \mathbb{1}_{k+1\dots N},$$

where

$$\mathcal{L}_1(\rho) = |0\rangle\langle 1| \rho |1\rangle\langle 0| - \frac{1}{2}\{\rho, |1\rangle\langle 1|\}$$

is an amplitude damping process on a single qubit, preparing the state  $|0\rangle$ . Then  $e^{t\mathcal{L}_N} = (e^{t\mathcal{L}_1})^{\otimes N}$  has gap  $1/2$  and mixing time of order  $O(\log N)$  [27]. The fixed point is of course the pure state  $|0\dots 0\rangle\langle 0\dots 0|$ .

Now consider  $\mathcal{L}_1^\varepsilon$ , a rotation of  $\mathcal{L}_1$ , which prepares the state  $|\alpha_0\rangle = \sqrt{1-\varepsilon^2}|0\rangle + \varepsilon|1\rangle$ . We have  $\|\mathcal{L}_1 - \mathcal{L}_1^\varepsilon\| = O(\varepsilon)$ , but the new fixed point  $|\alpha_0\rangle\langle\alpha_0|^{\otimes N}$  is almost orthogonal to the original one, since the overlap between the two is

$$\langle 0\dots 0|\alpha_0\dots\alpha_0\rangle = \langle 0|\alpha_0\rangle^N = (1-\varepsilon^2)^{N/2} \sim e^{-N\varepsilon^2/2} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

This shows that in general the dependence on the support of the observable in equation (11) cannot be improved.

#### 4.4 Do we need all the assumptions?

Are the assumptions of theorem 4.6 a reasonable set? We have just shown that we must necessarily consider local observables if we are to have meaningful bounds, but what about the other conditions? We will now present, in order, an example of each of the following:

1. verifies frustration-freeness, truncated rapid mixing, global rapid mixing, and belongs to a uniform family, but does not have a unique fixed point and fails to verify LTQO;
2. verifies frustration-freeness, LTQO, and has a unique fixed point, but fails to verify global rapid mixing, truncated rapid mixing, and does not belong to a uniform family;
3. (presented in Appendix A) verifies frustration-freeness, LTQO, has a unique fixed point, and verifies global rapid mixing, but it neither verifies truncated rapid mixing, nor belongs to a uniform family.

All these systems will be shown to be unstable.

*Example 4.12.* Consider a 1D chain composed of  $N$  4-level systems, with an independent Lindbladian acting on each site, having the following Lindblad operators

$$L_1 = |0\rangle\langle 1|, \quad L_2 = |0\rangle\langle 3|, \quad L_3 = |2\rangle\langle 1|, \quad L_4 = |2\rangle\langle 3|,$$

---

<sup>6</sup>Indeed, all global stability results for quantum Lindbladians we are aware of have a dependency on the total Hilbert space dimension [26, 45].

and call

$$\mathcal{L}_0(\rho) = \sum_{i=1}^4 L_i \rho L_i^* - \frac{1}{2} \{\rho, L_i^* L_i\}.$$

The global Lindbladian  $\mathcal{L}_N$  is given by applying  $\mathcal{L}_0$  independently on each site  $k = 1 \dots N$ :

$$\mathcal{L}_N = \sum_{k=1}^N \mathbb{1}_{1,\dots,k-1} \otimes \mathcal{L}_0 \otimes \mathbb{1}_{k+1,\dots,N}.$$

Then we have that

$$\mathcal{L}_0(|i\rangle\langle j|) = \begin{cases} |0\rangle\langle 0| + |2\rangle\langle 2| - 2|i\rangle\langle j| & \text{if } i = j, i, j \in \{1, 3\} \\ 0 & \text{if } i = j, i, j \in \{0, 2\} \\ -[\chi_{\{1,3\}}(i) + \chi_{\{1,3\}}(j)]|i\rangle\langle j| & \text{if } i \neq j. \end{cases}$$

Diagonal states of the form  $|i\rangle\langle i|$  evolve according to the classical Markov process embedded in the Lindbladian, while off-diagonal elements  $|i\rangle\langle j|$  evolve as

$$T_t(|i\rangle\langle j|) = \exp(-t[\chi_{\{1,3\}}(i) + \chi_{\{1,3\}}(j)])|i\rangle\langle j|.$$

This implies that  $\mathcal{F}_{\mathcal{L}_0} = \text{span}\{|0\rangle\langle 0|, |2\rangle\langle 2|, |0\rangle\langle 2|, |2\rangle\langle 0|\}$ . Since  $\mathcal{L}_0$  has gap equal to 1, theorem 6.1 below implies that  $\mathcal{L}_N$  verifies both truncated and global rapid mixing.  $\mathcal{L}_N$  is frustration-free and forms a uniform family, but it does not verify either the unique fixed point condition, nor LTQO: clearly, the fixed points  $|0 \dots 0\rangle\langle 0 \dots 0|$  and  $|2 \dots 2\rangle\langle 2 \dots 2|$  can be distinguished locally.

Consider now the following additional Lindbladian

$$\mathcal{E}_0(\rho) = \frac{2}{N} \left[ |0\rangle\langle 2| \rho |2\rangle\langle 0| - \frac{1}{2} \{\rho, |2\rangle\langle 2|\} \right].$$

Then, we have:

$$(\mathcal{L}_0 + \mathcal{E}_0)(|i\rangle\langle j|) = \begin{cases} |0\rangle\langle 0| + |2\rangle\langle 2| - 2|i\rangle\langle j| & \text{if } i = j, i, j \in \{1, 3\} \\ \frac{2}{N}(|0\rangle\langle 0| - |i\rangle\langle j|) & \text{if } i = j = 2 \\ 0 & \text{if } i = j = 0 \\ -\left[\chi_{\{1,3\}}(i) + \chi_{\{1,3\}}(j) + \frac{\chi_{\{i,j\}}(2)}{N}\right]|i\rangle\langle j| & \text{if } i \neq j. \end{cases}$$

Again, this implies that  $\mathcal{F}_{\mathcal{L}_0 + \mathcal{E}_0} = \{|0\rangle\langle 0|\}$ . Consequently  $\mathcal{L}_N + \mathcal{E}_N$  verifies LTQO, frustration-freeness, and has a unique fixed point. It is not a uniform family, and it clearly does not verify truncated rapid mixing or global rapid mixing, as it is not even globally gapped:

$$\mathcal{L}_N + \mathcal{E}_N(|2, 0 \dots 0\rangle\langle 2, 0 \dots 0|) = -\frac{2}{N}(|0, 0 \dots 0\rangle\langle 0, 0 \dots 0| - |2, 0 \dots 0\rangle\langle 2, 0 \dots 0|).$$

Analogously,  $\mathcal{L}_N + \mathcal{E}_N^*$  verifies the same conditions as  $\mathcal{L}_N + \mathcal{E}_N$ , but the unique fixed point is now  $|2\rangle\langle 2|$ .

All these three systems are unstable, since we can transform one into the other by applying a perturbation of order  $O(1/N)$ , yet the fixed points of  $\mathcal{L}_N + \mathcal{E}_N$  and  $\mathcal{L}_N + \mathcal{E}_N^*$  are locally orthogonal (while  $\mathcal{L}_N$  has both of them as fixed points).

## 5 Toolbox for the proof

The proof of theorem 4.6 will be done in two steps. First, we will introduce a new condition, called *local rapid mixing*, and prove that the hypotheses of theorem 4.6 imply local rapid mixing. The second step, independent of the first, will consist in proving that local rapid mixing and uniqueness of the fixed point imply the desired stability result. But first, we will need to introduce some useful tools.

### 5.1 Lieb-Robinson bounds for Lindbladian evolution

We first recall a generalization of Lieb-Robinson bounds to non-Hamiltonian evolution, due to [42] and [40], which we will use to derive a number of useful tools that allow us to approximate the support of an evolving observable with a finite set which grows linearly in time.

**Theorem 5.1.** *Let  $\mathcal{L} = \sum_{u,r} \mathcal{L}_u(r)$  be a local Lindbladian satisfying the following condition for some positive  $\mu, v$*

$$\sum_{x \in \Lambda} \sum_{r \geq \text{dist}(u,x)} \|\mathcal{L}_x(r)\|_{1 \rightarrow 1, cb} |b_x(r)| e^{\mu r} \leq \frac{v}{2} < \infty \quad \forall u \in \Lambda. \quad (12)$$

(The condition is verified automatically if  $\mathcal{L}$  has finite-range or exponentially decaying interactions.)

Let  $O_X$  be an observable supported on  $X \subset \Lambda$ , and denote by  $O_X(t) = T_t^*(O_X)$  its evolution under  $\mathcal{L}$ . Let  $K : \mathcal{A}_Y \rightarrow \mathcal{A}_Y$  be a super-operator supported on  $Y \subset \Lambda$  which vanishes on  $\mathbb{1}$ . Then it holds that

$$\|K(O(t))\| \leq \|K\|_{\infty \rightarrow \infty, cb} \|O_X\| C(X, Y) e^{-\mu \text{dist}(X, Y)} (e^{vt} - 1), \quad (13)$$

where  $C(X, Y) = \min(|X|, |Y|)$ .  $v$  is called the Lieb-Robinson speed (or velocity) of  $\mathcal{L}$ .

Note that if  $\mathcal{L}$  has interaction strength  $(J, f(r))$ , then equation (12) can be replaced with

$$\sum_{x \in \Lambda} \sum_{r \geq \text{dist}(u,x)} f(r) |b_x(r)| e^{\mu r} \leq \frac{v}{2J} < \infty \quad \forall u \in \Lambda. \quad (14)$$

This implies that, if equation (14) is verified, different Lindbladians whose strengths can be uniformly bounded will have uniform Lieb-Robinson velocities.

From now on, we will only consider Lindbladians which verify equation (12).

**Lemma 5.2** (Comparing different dynamics). *Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be two local Lindbladians, having strengths bounded by  $(J, f(r))$ . Let  $Y = \text{supp}(\mathcal{L}_1 - \mathcal{L}_2)$  be the set on which the two differ. Consider  $O_X$  an operator supported on  $X \subset \Lambda$ , and call  $O_i(t)$  its evolution under  $\mathcal{L}_i$ . Then it holds that*

$$\|O_1(t) - O_2(t)\| \leq \|\mathcal{L}_1 - \mathcal{L}_2\|_{1 \rightarrow 1, cb} \|O_X\| C(X, Y) e^{-\mu \text{dist}(X, Y)} \left( \frac{e^{vt} - 1 - vt}{v} \right). \quad (15)$$

*Proof.* Let  $h(t) = O_1(t) - O_2(t)$ . Calculating its derivative, we obtain

$$h'(t) = \mathcal{L}_1^* O_1(t) - \mathcal{L}_2^* O_2(t) = \mathcal{L}_1^* h(t) + (\mathcal{L}_1^* - \mathcal{L}_2^*) O_2(t).$$

Since  $h(0) = 0$ , this differential equation for  $h(t)$  has solution

$$h(t) = O_1(t) - O_2(t) = \int_0^t e^{(t-s)\mathcal{L}_1^*} (\mathcal{L}_1^* - \mathcal{L}_2^*) O_2(s) ds,$$

giving us a useful integral representation for  $O_1(t) - O_2(t)$ . From this, we obtain the estimate

$$\|h(t)\| \leq \int_0^t \|(\mathcal{L}_1^* - \mathcal{L}_2^*) O_2(s)\| ds,$$

where we have used the fact that  $e^{t\mathcal{L}_1^*}$  is a contraction with respect to  $\|\cdot\|_\infty$  for each  $t \geq 0$ .

We can now apply the Lieb-Robinson bound (equation (13)) to the previous estimate, to obtain

$$\|h(t)\| \leq \|\mathcal{L}_1 - \mathcal{L}_2\|_{1 \rightarrow 1, cb} \|O_X\| C(X, Y) e^{-\mu \text{dist}(X, Y)} \int_0^t (e^{vs} - 1) ds,$$

which implies the claimed bound.  $\square$

**Lemma 5.3** (Localizing the evolution). *Let  $O_A$  be an observable supported on finite  $A \subset \Lambda$ . Denote by  $O_A(t) = T_t^*(O_A)$  its evolution under a local Lindbladian  $\mathcal{L}$  with strength  $(J, f(r))$ . Given  $r > 0$ , denote by  $O_r(t)$  its evolution under the localized Lindbladian  $\mathcal{L}_{A(r)}$ , where  $A(r) = \{x \in \Lambda \mid \text{dist}(x, A) \leq r\}$ .*

*Then it holds that*

$$\|O_A(t) - O_r(t)\| \leq \|O_A\| |A| \frac{J}{v} (e^{vt} - 1 - vt) g(r) \quad (16)$$

where  $g(r)$  is an exponentially decaying function.

*Proof.* First, let us decompose  $O_A(t) - O_r(t)$  as a telescopic sum

$$O_A(t) - O_r(t) = \sum_{l \geq r} O_{l+1}(t) - O_l(t).$$

Each element in the sum is the difference between the evolution generated by  $\mathcal{L}_{A(l)}$  and  $\mathcal{L}_{A(l+1)}$ . Since they are restrictions on different subsets of the same global Lindbladian, it is easy to explicitly write the difference

$$\mathcal{L}_{A(l+1)} - \mathcal{L}_{A(l)} = \sum_{\delta=0}^{l+1} \sum_{\text{dist}(u, A)=\delta} \mathcal{L}_u(l+1-\delta).$$

We can then apply our previous integral representation for  $O_{l+1}(t) - O_l(t)$ , and obtain:

$$\begin{aligned} O_A(t) - O_r(t) &= \sum_{l \geq r} \int_0^t e^{(t-s)\mathcal{L}_{A(l+1)}^*} (\mathcal{L}_{A(l+1)}^* - \mathcal{L}_{A(l)}^*) O_l(s) ds \\ &= \sum_{l \geq r} \sum_{\delta=0}^{l+1} \sum_{\text{dist}(u, A)=\delta} \int_0^t e^{(t-s)\mathcal{L}_{A(l+1)}^*} \mathcal{L}_u^*(l+1-\delta) O_l(s) ds. \end{aligned}$$

Now take norms, recalling that each  $e^{t\mathcal{L}_{A(l)}}$  is a contraction:

$$\|O_A(t) - O_r(t)\| \leq \sum_{l \geq r} \sum_{\delta=0}^{l+1} \int_0^t \|\mathcal{L}_u^*(l+1-\delta)O_l(s)\| ds.$$

Since  $d = \text{dist}(A, b_u(l+1-\delta)) = \max\{0, 2\delta-l-1\}$ , we can apply the Lieb-Robinson bound (equation (13)) to each of the norms inside the integrals, and obtain

$$\int_0^t \|\mathcal{L}_u^*(l+1-\delta)O_l(s)\| ds \leq Jf(l+1-\delta) \|O_A\| |A| e^{-\mu d} \left( \frac{e^{vt}-1}{v} - t \right).$$

We can group the terms in the sum by their distance from  $A$ . Denoting  $q(l) = |\{u : \text{dist}(u, A) = l\}|$ , we have

$$\begin{aligned} & \|O_A(t) - O_r(t)\| \\ & \leq \|O_A\| |A| J \frac{e^{vt}-vt-1}{v} \sum_{l \geq r} \sum_{\delta=0}^{\frac{l+1}{2}} q(\delta) f(l+1-\delta) + \sum_{\delta=\frac{l+1}{2}}^{l+1} q(\delta) f(l+1-\delta) e^{-\mu(2\delta-l-1)} \\ & \leq \|O_A\| |A| J \frac{e^{vt}-vt-1}{v} \sum_{l \geq r} \sum_{\delta=0}^{l+1} e^{-\mu\delta} q(\delta) f(l+1-\delta) e^{\mu(l+1-\delta)}. \end{aligned}$$

Since  $f(\delta)$  verifies (14),  $f(\delta)e^{\mu\delta}$  is still exponentially decaying. The same holds for  $q(\delta)e^{-\mu\delta}$ , since  $q(\delta)$  grows polynomially. In turn, this implies that

$$\tilde{g}(l) = \sum_{\delta=0}^{l+1} e^{-\mu\delta} q(\delta) f(l+1-\delta) e^{\mu(l+1-\delta)}$$

is the convolution of two exponentially decaying functions, and consequently is still exponentially decaying. This completes the proof, with  $g(r) = \sum_{l \geq r} \tilde{g}(l)$ .  $\square$

## 5.2 LTQO and frustration freeness

If  $\mathcal{L}$  verifies LTQO and frustration freeness, then local observables are not able to distinguish between a global periodic point and a local one.

**Lemma 5.4.** *Let  $O_A \in \mathcal{A}_A$  be an observable,  $\rho_0 \in \mathcal{X}_{\mathcal{L}}$ ,  $\rho_1 \in \mathcal{X}_{\mathcal{L}_{A(\ell)}}$ . Then*

$$\text{tr}|\rho_0 O_A - \rho_1 O_A| \leq \|O_A\| k(|A|) \Delta_0(\ell). \quad (17)$$

*Proof.* Frustration-freeness implies that  $\rho_0 \in \mathcal{X}_{\mathcal{L}_{A(\ell)}}$ , and thus by LTQO we have

$$\|\rho_0^A - \rho_1^A\|_1 \leq k(|A|) \Delta_0(\ell),$$

which implies the desired result.  $\square$

## 5.3 Local rapid mixing

Local rapid mixing is the intermediate step we are going through in order to prove theorem 4.6.

**Definition 5.5** (Local rapid mixing). Take  $A \subset \Lambda$ , and define the contraction of  $T_t$  relative to  $A$  as

$$\begin{aligned}\eta^A(T_t) &= \sup_{\substack{\rho \geq 0 \\ \text{tr } \rho = 1}} \sup_{\substack{O_A \in \mathcal{A}_A \\ \|O_A\|=1}} d_{\text{tr}}(T_t(\rho)O_A, T_t T_\phi(\rho)O_A) \\ &= \sup_{\substack{\rho \geq 0 \\ \text{tr } \rho = 1}} d_{\text{tr}}(\text{tr}_{A^c} T_t(\rho), \text{tr}_{A^c} T_t T_\phi(\rho)).\end{aligned}\tag{18}$$

We say that  $\mathcal{L}$  verifies local rapid mixing if, for each  $A \subset \Lambda$ , we have that

$$\eta^A(T_t) \leq k(|A|)g(t),\tag{19}$$

where  $k(r)$  grows polynomially in  $r$  and  $g(t)$  is a fast-decaying function.

*Observation 5.6.* It follows from the definition that  $\eta^A(T_t) \leq \eta^B(T_t)$  whenever  $A \subset B$ . In particular,  $\eta^A(T_t) \leq \eta(T_t)$ .

Note that, in contrast with definitions 4.3 and 4.4, the quantity  $\eta^A(T_t)$  depends on the evolution *on the whole system*, and not just on the subset  $A$ . Thus local rapid mixing is a very strong condition: the term  $k(r)$  appearing in equation (19) only depends on the support of  $A$ , and so the reduced mixing time (i.e. the time it takes for the reduced density matrix on the subset  $A$  to converge) is independent of system size. On the other hand, the requirement on the convergence rate is weaker; it must be faster than any polynomial, but not necessary exponential.

*Example 5.7.* A simple dissipative system verifying definition 5.5 is the tensor product of amplitude damping channels acting (with the same rate) on different qubits. Note that, though it might seem a trivial example, there are interesting dissipative systems of this form: among others, dissipative preparation of graph states [27] can be brought into this form by a non-local unitary rotation (which of course does not change the convergence rates).

Interestingly, if  $\mathcal{L}$  verifies local rapid mixing, then its fixed points have a particular character: they verify a fast-decay of correlations, meaning that the correlations between two spatially separated observables decays in distance faster than any polynomial. More specifically, if  $\mathcal{L}$  verifies local rapid mixing with  $g(t)$  as a decaying function, then the correlations in its fixed points decay as fast as  $g(t)$ . We will suppose for simplicity that  $T_t$  do not have periodic points beyond the fixed points (i.e.  $\mathcal{X}_{\mathcal{L}} = \mathcal{F}_{\mathcal{L}}$ ) and that its fixed point is unique.

**Proposition 5.8.** Let  $O_A$  and  $O_B$  be operators supported on regions  $A$  and  $B$ , with  $\|O_A\| = \|O_B\| = 1$ , and let  $d_{AB}$  be the distance between them. Define the correlation function

$$\text{corr}(O_A, O_B) = |\langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle|,$$

where  $\langle O \rangle = \text{tr}(\rho_\infty O)$  is the expectation value of the operator  $O$  acting on the fixed point  $\rho_\infty$ . Then we have that

$$\text{corr}(O_A, O_B) \leq p(|A| + |B|)g(d_{AB}),$$

where  $g(d)$  is a fast-decaying function and  $p(\cdot)$  is growing polynomially. Moreover, if  $\mathcal{L}$  verifies local rapid mixing with an exponential convergence rate, then  $g(d)$  is also exponentially decaying.



*Proof.* Let us call  $C = A \cup B$ . Fix a product state  $\rho_0$  (i.e.  $\rho_0 = \otimes |0\rangle\langle 0|$ ) and call  $\rho(t) = T_t(\rho_0)$ . From the definition of local contraction, we have that

$$\text{tr}[(\rho(t) - \rho_\infty)O_A O_B] \leq \eta^C(T_t) \leq k(|A| + |B|)g(t),$$

where we have used local rapid mixing to bound  $\eta^C$ .

We now use lemma 5.3 to localize the evolution:

$$\begin{aligned} \text{tr}[\rho(t)O_A O_B] &= \text{tr}[\rho_0 T_t^*(O_A O_B)] \leq \\ &\text{tr}\left[\rho_0 (T_t^{C(s)})^*(O_A O_B)\right] + |A \cup B| \frac{J}{v} h(t) f(s), \end{aligned}$$

where  $h(t)$  grows exponentially in time and  $f(s)$  is exponentially decaying. If  $s \leq d_{AB}/2$ , then  $T_t^{C(s)}$  is a product of terms acting on disjoint regions, and thus we have that  $T_t^{*,C(s)}(O_A O_B) = T_t^{*,A(s)}(O_A) T_t^{*,B(s)}(O_B)$ .

By applying lemma 5.3 and local rapid mixing one last time, since  $\rho_0$  is a product state we have that

$$\begin{aligned} \text{tr}\left[\rho_0 T_t^{*,A(s)}(O_A) T_t^{*,B(s)}(O_B)\right] \\ \leq \langle O_A \rangle \langle O_B \rangle + (|A| + |B|) \frac{J}{v} h(t) f(s) + [k(|A|) + k(|B|)]g(t). \end{aligned}$$

Putting everything together, for  $s \leq d_{AB}/2$  we have shown that

$$\text{corr}(O_A, O_B) \leq p(|A| + |B|)[g(t) + \frac{J}{v} h(t) f(s)] \quad \forall t \geq 0,$$

and by choosing a  $t = O(d_{AB})$  such that  $h(t)f(s)$  is exponentially decaying, we can finally bound the correlation as follows:

$$\text{corr}(O_A, O_B) \leq p(|A| + |B|)g'(d_{AB}),$$

where  $g'$  is fast-decaying. Note that if  $g(t)$  was exponentially decaying, then  $g'(d)$  is too.  $\square$

In the following sections, we will prove that the assumptions on the Lindbladians in the stability theorem 4.6 imply local rapid mixing. Together with the above result, this implies fast decay of correlations for local Lindbladians satisfying these assumptions, as stated in theorem 4.10.

## 6 Proof of main result

### 6.1 Step 1a: from truncated to local rapid mixing

In this section, we want to show that if  $\mathcal{L}$  verifies frustration-freeness, LTQO, and truncated rapid mixing, then it verifies local rapid mixing. This is half of the first step needed to prove theorem 4.6; the second half of this first step, involving global rapid mixing and uniform families of Lindbladians, will be presented in the next section.

Let us recall a result from [27].

**Theorem 6.1** (Contraction for commuting Liouvillians). *Let  $\{\mathcal{L}_j\}_{j=0}^n$  be a set of commuting Lindbladians. Define  $\mathcal{L} = \sum_j \mathcal{L}_j$  and the corresponding evolutions  $T_t^j = e^{t\mathcal{L}_j}$  and  $T_t = e^{t\mathcal{L}}$ . Then:*

$$\eta(T_t) \leq \sum_j \eta(T_t^j). \quad (20)$$

In particular, this means that for a local Lindbladian  $\mathcal{L} = \sum_u \mathcal{L}_u$  with finite-range interactions composed of commuting terms  $[\mathcal{L}_u, \mathcal{L}_v] = 0$ , and such that the local terms are uniformly gapped (i.e.  $\eta(T_t^u) \leq ce^{-\gamma t}$  for some  $c$  and  $\gamma$  independent of  $u$  and of the global system size), it holds that

$$\eta(T_t^A) \leq \sum_{u \in A} \eta(T_t^u) \leq c|A|e^{-\gamma t},$$

and thus  $T_t$  verifies truncated rapid mixing.

**Corollary 6.2.** *Let  $\mathcal{L}$  be a local Lindbladian, with finite-range interactions and commuting terms. If the local terms of  $\mathcal{L}$  are uniformly gapped, then  $\mathcal{L}$  verifies truncated rapid mixing.*

Interestingly, theorem 6.1 also implies that if equation (9) holds for balls, it also holds for disjoint unions of balls, since for any disjoint sets  $A$  and  $B$  the respective truncated Lindbladians commute:

$$A \cap B = \emptyset \Rightarrow [\mathcal{L}_A, \mathcal{L}_B] = 0,$$

and consequently  $\eta(T_T^{A \cup B}) \leq \eta(T_t^A) + \eta(T_t^B)$ . Thus verifying equation (9) for balls is enough.

We are now ready to prove the key lemma for this step of the proof.

**Lemma 6.3.** *Let  $\mathcal{L}$  verify LTQO and frustration-freeness. Then, for all  $A \subset \Lambda$  and all  $t \geq 0$ , we have*

$$\eta^A(T_t) \leq \frac{J}{v} |A| (e^{vt} - 1 - vt)g(s) + p(|A|)\Delta_0(s) + \eta^A(T_t^{A(s)}) \quad \forall s \geq 0, \quad (21)$$

where  $g(s)$  is an exponentially decaying function,  $p(\cdot)$  is growing polynomially and the constant  $v$  is the Lieb-Robinson speed for  $T_t$ .

*Proof.* Let  $O_A$  be an observable supported on  $A$  with  $\|O_A\| = 1$ . Fix  $s \geq 0$ , and call  $B = A(s)$ . Then, by the triangle inequality

$$\begin{aligned} \|T_t^*(O_A) - T_\phi^* T_t^*(O_A)\| &\leq \|T_t^*(O_A) - (T_t^B)^*(O_A)\| \\ &\quad + \|(T_t^B)^*(O_A) - (T_t^B T_\phi^B)^*(O_A)\| \\ &\quad + \|(T_t^B T_\phi^B)^*(O_A) - T_\phi^* T_t^*(O_A)\|. \end{aligned}$$

We can bound the first term in the sum using lemma 5.3 (which used Lieb-Robinson bounds to localize the evolution):

$$\|T_t^*(O_A) - (T_t^B)^*(O_A)\| \leq |A| \frac{J}{v} (e^{vt} - 1 - vt)g(s), \quad g \text{ exponentially decaying.}$$

The second term is bounded by the contraction of the localized evolution  $T_t^B$ :

$$\|(T_t^B)^*(O_A) - (T_t^B T_\phi^B)^*(O_A)\| \leq \eta^A(T_t^B).$$

To bound the third term, we rewrite it as follows:

$$\|(T_t^B T_\phi^B)^*(O_A) - T_\phi^* T_t^*(O_A)\| = \sup_\rho \text{tr} |[T_t^B T_\phi^B(\rho) - T_t T_\phi(\rho)] O_A|.$$

Observe that  $\mathcal{X}_\mathcal{L}$  (resp.  $\mathcal{X}_{\mathcal{L}_B}$ ) is closed under the action of  $T_t$  (resp.  $T_t^B$ ) (and the evolution on these spaces is unitary). This means that  $\rho_0 := T_t^B T_\phi^B(\rho) \in \mathcal{X}_{\mathcal{L}_B}$  and  $\rho_1 := T_t T_\phi(\rho) \in \mathcal{X}_\mathcal{L}$ . We can now apply lemma 5.4 (which used LTQO and frustration-freeness) to obtain

$$\text{tr} |(T_t^B T_\phi^B(\rho) - T_t T_\phi(\rho)) O_A| = \text{tr} |(\rho_1 - \rho_0) O_A| \leq p(|A|) \Delta_0(s),$$

which completes the proof.  $\square$

**Proposition 6.4** (From truncated to local rapid mixing). *Let  $\mathcal{L}$  verify LTQO and frustration-freeness. Then, if  $\mathcal{L}$  verifies truncated rapid mixing, it also verifies local rapid mixing.*

*Proof.* Fix  $A \subset \Lambda$  and  $s \geq 0$ . Remember that truncated rapid mixing is a bound on the contraction of the local evolution  $T_t^A$ :

$$\eta(T_t^A) \leq k(|A|) e^{-t\gamma}.$$

In definition 4.3 this is required to hold only for ball-shaped domains, and we know that thanks to theorem 6.1 it extends to disjoint unions of balls. The problem we face now, if we want to plug this inequality into the result of lemma 6.3, is that if  $A$  is not connected then  $A(s)$  is not of this form when  $s$  is bigger than the distance between the connected components of  $A$ ; instead, it has a “peanut-like” shape.

The key observation is that, when two disconnected components of  $A$  grow enough to “touch” each other, the ball containing both has size polynomial in the distance  $d$  between them, and thus asymptotically we do not lose anything by changing from  $A(d)$  to the smallest ball containing  $A(d)$ . While in general  $\eta(T_t^A)$  is not monotone as  $A$  increases, Lieb-Robinson bounds and lemma 5.4 are (in the sense that replacing  $A(d)$  by a ball containing it does not affect the resulting bounds – it would actually improve them), so it is enough to allow for discontinuous steps in the growth of  $A(s)$  in order to ensure it is a disjoint union of balls.

Given this, we have that for each  $s \geq 0$ :

$$\eta^A(T_t) \leq \frac{J}{v} |A| (e^{vt} - 1 - vt) g(s) + p(|A|) \Delta_0(s) + k(|A|) h(s) e^{-t\gamma},$$

where  $h(s)$  is bounded by a polynomial in  $s$ .

We want to show that, for each time  $t \geq 0$ , we can choose  $s = s(t)$  in such a way that both  $e^{vt} g \circ s(t)$  and  $e^{-t\gamma} h \circ s(t)$  have super-polynomial decay in  $t$ . For each  $t \geq 0$ , let us choose an  $s_0 = s_0(t)$  such that

$$(e^{vt} - 1 - vt) g(s) \leq e^{-vt} \quad \forall s \geq s_0.$$

We can choose  $s_0(t)$  such that it grows at most linearly:  $s_0(t) = O(t)$ . Let us choose a second  $s_1 = s_1(t)$  such that

$$h(s) e^{-\gamma t} \leq e^{-\frac{\gamma}{2} t} \quad \forall s \leq s_1.$$

We can choose  $s_1(t)$  such that  $\log(s_1(t)) = O(t)$ .

Since  $s_0(t)$  grows at most linearly in  $s$ , while  $s_1(t)$  grows at least exponentially in  $t$ , by choosing  $s = s_0(t)$  we have that, up to constants which do not depend on the system size,  $s(t) \leq s_1(t)$  for all  $t \geq 0$ .

With this choice of  $s(t)$ ,  $\Delta_0(s(t))$  is still fast-decaying in  $t$ , thus

$$\eta^A(T_t) \leq \frac{J}{v} |A| e^{-vt} + k(|A|) \Delta_0(s(t)) + p(|A|) e^{-t \frac{\gamma}{2}} \leq q(|A|) \varphi(t),$$

where  $\varphi(t)$  is a fast-decaying function of  $t$  and  $q(\cdot)$  grows polynomially. This completes the proof.  $\square$

## 6.2 Step 1b: from global to local rapid mixing

We will now focus on the other set of hypotheses of theorem 4.6, in which the truncated rapid mixing hypothesis is replaced by a simpler global rapid mixing condition and an assumption on the “self-similarity” of the family of Lindbladians being considered. To explain the main idea, consider a translationally-invariant model with open boundary conditions. Then the truncated dynamics  $T_t^{b_x(r)}$  is exactly the same as the global dynamics defined on the smaller lattice  $b_x(r)$ . This implies that, just by looking at how the global contraction scales with the lattice size, we can check the truncated rapid mixing assumption. This scaling of the mixing time with the system size is a more natural (and simpler) property to consider.

We would like to generalize this idea to models with arbitrary boundary conditions. We first need to pin down some minimal properties that the family of Lindbladians should satisfy, for the scaling of their mixing times to be meaningful.

**Definition 6.5.** *Consider a family of Lindbladians  $\{\mathcal{L}^L\}_{L \in \mathbb{N}}$  with finite range or exponentially decaying interactions, where each  $\mathcal{L}^L$  is defined on  $\Lambda_L \subset \mathbb{Z}^D$  such that  $b_0(L) \subset \Lambda_L$ . We say that the family is uniform if the following conditions hold:*

(i)  $\mathcal{L}^L$  are uniformly bounded in strength:

$$\sup_{L,u,r} \|\mathcal{L}_u^L(r)\|_{1 \rightarrow 1,cb} = J < \infty, \quad \sup_{L,u} \frac{\|\mathcal{L}_u^L(r)\|_{1 \rightarrow 1,cb}}{J} = f(r),$$

where  $f(r)$  is compactly supported (finite range interactions) or exponentially decaying.

(ii) There exists a translationally-invariant<sup>7</sup> Lindbladian  $\mathcal{M}$ , defined on  $\mathbb{Z}^D$ , such that for all  $L > 0$ :

$$\mathcal{L}_{\Lambda_L \setminus \partial \Lambda_L}^L = \mathcal{M}_{\Lambda_L \setminus \partial \Lambda_L},$$

where  $\partial \Lambda_L$  is defined as

$$\partial \Lambda_L = \Lambda_L \setminus b_0(L).$$

---

<sup>7</sup>We restrict to translationally-invariant Lindbladians for simplicity, and because this is the most important case. The results in fact extend to more general families of Lindbladians, for which  $M$  in definition 6.5 is not translationally-invariant.

In other words, the local terms of the Lindbladians cannot depend on the system size, up to boundary conditions (i.e. terms whose support intersects  $\partial\Lambda_L$ ), which are allowed to change freely between the different system sizes as long as they can be uniformly bounded in norm. This includes, for example, all translationally-invariant Lindbladians with periodic, open, or fixed boundary conditions.

Since we are interested in observables whose support is not connected, we will use the same convention laid out in the proof of proposition 6.4: with an abuse of notation, for a finite  $A \subset \mathbb{Z}^D$  we will denote by  $A(s)$  the smallest disjoint union of balls containing the set  $\{x | d(x, A) \leq s\}$ . As noted in the previous section,  $|A(s)|$  still grows polynomially with  $s$ , and this notation allows us to define the following:

**Definition 6.6.** Let  $A \subset \mathbb{Z}^D$  be a finite set, and let  $A(s) = \bigcup_{i=1}^k b_{x_i}(L_i)$ , for some  $x_i \in \mathbb{Z}^D$  and some  $L_i \geq 0$ . We will denote by  $\mathcal{L}^{A(s)}$  the formal sum

$$\mathcal{L}^{A(s)} = \sum_{i=1}^k \mathcal{L}^{L_i},$$

acting on  $\bigotimes_i \Lambda_{L_i}$ .

If we remove the boundaries of each  $\Lambda_{L_i}$ , then we can identify what is left with  $A(s)$ , by translating each  $b_0(L_i)$  to  $b_{x_i}(L_i)$ , and consequently we can identify  $\mathcal{L}_{A(s)}^{A(s)}$  (i.e. the interaction terms of  $\mathcal{L}^{A(s)}$  which do not touch any boundary) with  $\mathcal{M}_{A(s)}$ , which in turn coincides with  $\mathcal{L}_{A(s)}^L$  for all  $L \geq \text{diam } A(s)$ .

The following result is the analogue of lemma 6.3.

**Lemma 6.7.** Let  $\{\mathcal{L}^L\}_L$  be a uniform family of Lindbladians verifying LTQO and frustration-freeness. Let  $A \subset \mathbb{Z}^D$  be a finite region. Take  $s \geq 0$  and  $L \geq \text{diam } A(s)$ . Then for  $t \geq 0$ , we have that

$$\eta^A(T_t^L) \leq \frac{J}{v} |A| (e^{vt} - 1 - vt)g(s) + k(|A|)\Delta_0(s) + \eta^A\left(T_t^{A(\beta s)}\right) \quad \forall s \geq 0, \quad (22)$$

where  $g(s)$  is an exponentially decaying function,  $k(\cdot)$  grows polynomially, the constant  $v$  is the Lieb-Robinson speed for  $T_t^L = e^{t\mathcal{L}^L}$ ,  $T_t^{A(\beta s)}$  is the evolution generated by  $\mathcal{L}^{A(\beta s)}$ , and  $\beta > 1$  is a constant such that

$$|A(\beta s)| \geq 2|A(s)|.$$

*Proof.* Let  $O_A$  be an observable supported on  $A$  with  $\|O_A\| = 1$ . Call  $S_t$  the evolution generated by  $\mathcal{L}^{A(\beta s)}$  and  $V_t$  the one generated by  $\mathcal{L}_{A(s)}^{A(\beta s)} = \mathcal{L}_{A(s)}^L$ . For notational simplicity, we drop the superscript  $L$  from  $T_t^L$ . Because we are supposing that  $\mathcal{L}^L$  is a uniform family,  $V_t$  can be seen as the truncation of either  $T_t$  or  $S_t$ . And since  $\beta$  is such that  $|A(\beta s)| \geq 2|A(s)|$ , the  $1 \rightarrow 1$  norm of  $V_t - V_t V_\phi$  is the same in both cases, because of observation 3.1.

Following the same argument as in the proof of lemma 6.3, we have that:

$$\begin{aligned} & \|T_t^*(O_A) - T_t T_\phi^*(O_A)\| \\ & \leq \frac{J}{v} (e^{vt} - vt - 1)g(s) + p(|A|)\Delta_0(s) + \|V_t^*(O_A) - V_t^* V_\phi^*(O_A)\|. \end{aligned} \quad (23)$$

On the other hand, by the triangle inequality we have:

$$\begin{aligned}\|V_t^*(O_A) - V_t^*V_\phi^*(O_A)\| &\leq \|V_t^*(O_A) - S_t^*(O_A)\| \\ &\quad + \|S_t^*S_\phi^*(O_A) - V_t^*V_\phi^*(O_A)\| \\ &\quad + \|S_t^*(O_A) - S_t^*S_\phi^*(O_A)\|.\end{aligned}$$

Since  $V_t$  is a truncation of  $S_t$ , we can bound the first term of the r.h.s. by lemma 5.3, the second term using LTQO, and the third one simply with the contraction of  $S_t$ .

$$\|V_t^*(O_A) - V_t^*V_\phi^*(O_A)\| \leq \frac{J}{v}(e^{vt} - vt - 1)g(s) + p(|A|)\Delta_0(s) + \eta^A(S_t). \quad (24)$$

Putting together equations (23) and (24), we get the desired bound.  $\square$

Finally, we have the analogous result to proposition 6.4. We omit the proof, since it follows the same steps as the proof of proposition 6.4, using lemma 6.7 instead of lemma 6.3.

**Proposition 6.8** (From global to local rapid mixing).

*Let  $\{\mathcal{L}^L\}_L$  be a uniform family of Lindbladians verifying LTQO and frustration-freeness. Then, if it verifies global rapid mixing,<sup>8</sup> it also verifies local rapid mixing.*

### 6.3 LTQO and unique fixed point

In the hypotheses of theorem 4.6, we did not assume LTQO for the case of uniform families of Lindbladians, arguing that it was not needed. We will now show that LTQO is already implied by the other assumptions in the theorem. For open boundary conditions this is trivial; in this case the truncated dynamics are themselves in the uniform family, and the unique fixed point assumption implies that there are no states to distinguish in definition 4.2. However, for other boundary conditions the result is non-trivial, and follows from the global rapid mixing condition.

**Theorem 6.9.** *Let  $\{\mathcal{L}^L\}_L$  be a uniform family verifying global rapid mixing and frustration-freeness, and suppose each  $\mathcal{L}^L$  has a unique fixed point and no other periodic points. Then  $\mathcal{L}^L$  verifies LTQO.*

*Proof.* Fix a finite set  $A \subset \mathbb{Z}^D$ . Let  $s \geq 0$ , and fix  $L \geq \text{diam } A(s)$ . Let us denote by  $S_t$  the evolution generated by  $\mathcal{L}^{A(s)}$  as in definition 6.6, and  $V_t = \exp(t\mathcal{L}_{A(s)}^{A(s)})$ . Call  $\rho_\infty$  the unique fixed point of  $\mathcal{L}^{A(s)}$ . Consider a periodic point  $\rho_0$  for  $\mathcal{L}_{A(s)}^{A(s)}$ . Then by lemma 5.2, for each observable  $O_A$  supported on  $A$ , we have that

$$\|V_t^*(O_A) - S_t^*(O_A)\| \leq \|O_A\| |A| p(s) e^{-\mu s} e^{vt},$$

where  $p(s)$  grows polynomially in  $s$ . This implies that, by calling  $\rho_1 = V_t^{-1}(\rho_0)$ :

$$\text{tr}[O_A(\rho_0 - S_t(\rho_1))] = \text{tr}[O_A[V_t(\rho_1) - S_t(\rho_1)]] \leq \|O_A\| |A| p(s) e^{-\mu s} e^{vt}.$$

<sup>8</sup>If we allow non-translationally-invariant Lindbladians in definition 6.5, we can no longer argue that rapid mixing holds for arbitrary balls by translation. Instead, we must require global rapid mixing for all Lindbladians obtained by truncating  $M$  to an arbitrary finite region and adding boundary terms.

On the other hand, global rapid mixing implies that

$$\mathrm{tr}[O_A(S_t(\rho_1) - \rho_\infty)] \leq \|O_A\| \eta(S_t) \leq \|O_A\| q(s) |A| e^{-\gamma t},$$

with  $q(s)$  growing polynomially. Then, by choosing  $vt = \mu s/2$ , we have that

$$\Delta_0(s) = |A| p(s) e^{-\mu s} e^{vt} + q(s) |A| e^{-\gamma t}$$

is a fast-decaying function, and

$$\mathrm{tr}[O_A(\rho_0 - \rho_\infty)] \leq \|O_A\| \Delta_0(s).$$

Since  $\rho_0$  was an arbitrary periodic point of  $\mathcal{L}_{A(s)}^{A(s)}$ , by the triangle inequality (comparing each local fixed point with  $\rho_\infty$ ) we obtain LTQO for  $\mathcal{L}^{A(s)}$ :

$$\|\rho_0^A - \rho_\infty^A\|_1 \leq \Delta_0(s) \quad \forall \rho_0 \in \mathcal{X}_{V_t}.$$

Now let  $\rho$  be a fixed point of  $\mathcal{L}_{A(s)}^L$ . By frustration freeness, we have that  $\rho^{A(s)} = \mathrm{tr}_{A(s)^c} \rho$  is a fixed point of  $\mathcal{L}_{A(s)}^L = \mathcal{L}_{A(s)}^{A(s)}$ , and thus

$$\|\rho^A - \rho_\infty^A\|_1 \leq \Delta_0(s).$$

□

Interestingly, a similar proof also holds when we replace global rapid mixing with local rapid mixing.

**Theorem 6.10.** *Let  $\mathcal{L}$  verify local rapid mixing, and suppose it has a unique fixed point. Then  $\mathcal{L}$  verifies LTQO.*

*Proof.* Let  $\rho_\infty$  be the unique global fixed point of  $\mathcal{L}$ . Consider a region  $A \subset \Lambda$ , and consider two fixed points  $\rho_0, \rho_1$  of  $\mathcal{L}_{A(\ell)}$ . To prove that  $\rho_0$  and  $\rho_1$  are indistinguishable on  $A$ , we will prove that both are indistinguishable from  $\rho_\infty^A = \mathrm{tr}_{A^c} \rho_\infty$ , and by triangle inequality

$$\|\rho_0^A - \rho_1^A\|_1 \leq \|\rho_0^A - \rho_\infty^A\|_1 + \|\rho_1^A - \rho_\infty^A\|_1.$$

Let  $O_A$  be an observable supported on  $A$  such that  $\|O_A\| = 1$ . Fix  $\rho \in \mathcal{X}_{A(\ell)}$ . We want to find a fast-decaying function  $\Delta_0(\ell)$  such that

$$|\mathrm{tr}[O_A(\rho - \rho_\infty)]| \leq p(|A|) \Delta_0(\ell),$$

for some polynomially growing function  $p(\cdot)$ .

For each  $t \geq 0$  denote  $T_t = \exp(t\mathcal{L})$  and  $S_t = \exp(t\mathcal{L}_{A(\ell)})$ . Call  $\rho_1 = S_t^{-1}(\rho)$ . Then, by lemma 5.3, it holds that

$$|\mathrm{tr}[O_A(T_t(\rho_1) - \rho)]| = |\mathrm{tr}[O_A(T_t(\rho_1) - S_t(\rho_1))]| \leq c |A| e^{vt} f(\ell),$$

where  $f(\ell)$  is an exponentially decaying function.

On the other hand, local rapid mixing implies that

$$|\mathrm{tr}[O_A(T_t(\rho_1) - \rho_\infty)]| = |\mathrm{tr}[O_A(T_t(\rho_1) - T_\infty(\rho_1))]| \leq \eta^A(T_t) = k(|A|)g(t),$$

where  $g(t)$  is fast-decaying in  $t$ .

Putting these last two bounds together, we have that for all  $t \geq 0$ :

$$\begin{aligned} |\text{tr}[O_A(\rho - \rho_\infty)]| &\leq |\text{tr}[O_A(\rho - T_t \circ S_t^{-1}(\rho))]| + |\text{tr}[O_A(T_t \circ S_t^{-1}(\rho) - \rho_\infty)]| \\ &\leq c|A|e^{vt}f(\ell) + k(|A|)g(t). \end{aligned}$$

By choosing  $t = t(\ell) = O(\ell)$  such that  $|A|e^{vt}f(\ell)$  is decaying in  $\ell$ , we have proved the desired result by defining

$$\Delta_0(\ell) = c|A|e^{vt(\ell)}f(\ell) + k(|A|)g(t(\ell)).$$

□

## 6.4 Step 2: from local rapid mixing to stability

We now prove that local rapid mixing alone implies stability. This is the last step in the proof of theorem 4.6, as we already proved in the previous sections that the hypotheses of theorem 4.6 imply local rapid mixing. However, the following result also stands independently: if a system can be shown to verify local rapid mixing by other means, it will also be stable.

**Theorem 6.11.** *Let  $\mathcal{L}$  be a local Lindbladian verifying local rapid mixing, and having a unique fixed point  $\rho_\infty$  such that*

$$T_\phi = T_\infty = |\rho_\infty\rangle\langle\mathbb{1}|.$$

*Then, using the same notation as in theorem 4.6, for all observables  $O_A$  supported on  $A \subset \Lambda$  we have that*

$$\|O_0(t) - O_1(t)\| \leq c(|A|) \|O_A\| \mathcal{E}, \quad (25)$$

*for some  $c$  not depending on the system size, independent of  $t$ , and polynomial in  $|A|$ .*

*Proof.* Call  $T_t = e^{t\mathcal{L}}$  and  $S_t = e^{t(\mathcal{L}+E)}$ , and recall that  $O_1 = e^{t\mathcal{L}^*}(O_A)$  and  $O_2 = e^{t(\mathcal{L}^*+E^*)}(O_A)$ . Let us write the difference  $O_0 - O_1$  using the integral representation

$$O_0(t) - O_1(t) = \int_0^t S_{t-s}^* E^* T_s^*(O_A) ds = \sum_u \sum_r \int_0^t S_{t-s}^* E_u(r)^* T_s^*(O_A) ds,$$

which implies

$$\|O_0(t) - O_1(t)\| \leq \sum_u \sum_r \int_0^t \|E_u(r)^* O_0(s)\| ds,$$

where we used the fact that  $S_t$  is a contraction.

Fix a site  $u$  and a positive  $r$ , and call  $\delta = \text{dist}(A, b_u(r))$ . We can split the integral at a time  $t_0$  (to be fixed later, depending on  $\delta$ ). We bound the first part of the integral with Lieb-Robinson bounds:

$$\int_0^{t_0} \|E_u(r)^* O_0(s)\| ds \leq \mathcal{E} f(r) \|O_A\| |A| e^{-\mu\delta} \frac{e^{vt_0} - vt_0 - 1}{v}.$$



Now pick  $t_0 = t_0(\delta)$  such that

$$e^{-\mu\delta} \frac{e^{vt_0} - vt_0 - 1}{v} \leq e^{-\frac{\mu\delta}{2}}.$$

We can choose  $t_0(\delta) = \frac{\mu}{2} \frac{\log v}{v} \delta = O(\delta)$ .

If  $t \leq t_0(\delta)$ , then we have bounded the entire integral, and we are done. Otherwise, we treat the second part of the integral as follows:

$$\begin{aligned} \int_{t_0(\delta)}^t \|E_u(r)^* O_0(s)\| ds &= \int_{t_0(\delta)}^t \|E_u(r)^* (O_0(s) - T_\infty^*(O_A))\| ds \\ &\leq \mathcal{E} f(r) \|O_A\| \int_{t_0(\delta)}^\infty \eta^A(T_s) ds \\ &\leq \mathcal{E} f(r) \|O_A\| k(|A|) \int_{t_0(\delta)}^\infty g(s) ds, \end{aligned}$$

where we used

$$E_u^*(r) T_\infty^*(O_A) = E_u^*(r) |\mathbb{1}\rangle\langle\rho_\infty| |O_A\rangle\rangle = \langle\rho_\infty| O_A\rangle E_u^*(r) |\mathbb{1}\rangle = 0$$

together with the local rapid mixing hypothesis.

Since  $t_0(\delta)$  is linear in  $\delta$ , we have that

$$\tilde{g}(\delta) = e^{-\frac{\mu\delta}{2}} + \int_{t_0(\delta)}^\infty g(s) ds$$

is still a fast-decaying function. Putting the integral back together, we obtain

$$\int_0^t \|E_u(r)^* O_0(s)\| ds \leq \mathcal{E} \|O_A\| k_1(|A|) f(r) \tilde{g}(\delta).$$

Returning to the sum, we have proved that

$$\|O_0(t) - O_1(t)\| \leq k_1(|A|) \mathcal{E} \|O_A\| \sum_u \sum_r f(r) \tilde{g}(\text{dist}(A, b_r(u))).$$

It suffices to show that the sum on the r.h.s. is finite (and independent of system size). Let us decompose the sum as follows

$$\begin{aligned} &\sum_u \sum_r f(r) \tilde{g}(\text{dist}(A, b_r(u))) \\ &= \sum_{\text{dist}(u,A)=0} \sum_r f(r) \tilde{g}(0) + \sum_{d>0} \sum_{\text{dist}(u,A)=d} \left( \sum_{r=0}^d f(r) \tilde{g}(d-r) + \sum_{r=d+1}^\infty f(r) \tilde{g}(0) \right) \\ &= \tilde{g}(0) |A| \sum_r f(r) + \sum_{d>0} q(d) \left( \sum_{r=0}^d f(r) \tilde{g}(r-d) + \tilde{g}(0) \sum_{r=d+1}^\infty f(r) \right), \end{aligned}$$

where  $q(d) = |\{u : \text{dist}(u, A) = d\}|$  grows polynomially in  $d$ .

The first term is clearly bounded, since  $f(r)$  is summable. Since  $f$  and  $\tilde{g}$  are both fast-decaying functions, their discrete convolution  $f \star \tilde{g}(d) = \sum_{r=0}^d f(r)\tilde{g}(r-d)$  is also fast-decaying, and consequently summable against any polynomial. The same holds for  $\sum_{r>d} f(r)$ . This proves that the second term is also bounded. Calling

$$c(|A|) = k_1(|A|) \left[ \tilde{g}(0)|A| \sum_r f(r) + \sum_{d>0} q(d) \left( f \star \tilde{g}(d) + \tilde{g}(0) \sum_{r>d} f(r) \right) \right],$$

we can finally derive the claimed bound.  $\square$

## 7 Glauber dynamics

As an example of a non-trivial dynamics for which we can now prove stability using our results, we turn to one of the most studied dynamics in classical statistical mechanics: Glauber dynamics, a Markov process that samples thermal states of local (classical) Hamiltonians on lattices. Apart from being an interesting model in itself, it has important applications in Monte-Carlo Markov chain algorithms for numerical many-body physics [33]. Determining whether Glauber dynamics is stable against noise or errors is therefore an important – and, as far as we are aware, still open – question.

In this section, we present a natural embedding of Glauber dynamics into the Linbdlabian setting, showing how this embedded dynamics inherits properties from the classical Markov chain. We will then apply the results of section 4 to prove, in the appropriate regime, stability of Glauber dynamics.

We will consider a lattice spin system over  $\Gamma = \mathbb{Z}^D$  or  $\Gamma = (\mathbb{Z}/L\mathbb{Z})^D$ , with (classical) configuration space of a single spin a finite set  $S$ . For simplicity, we will consider the case  $S = \{+1, -1\}$ . For each  $\Lambda \subset \Gamma$ , we will denote by  $\Omega_\Lambda$  the space of configurations over  $\Lambda$ , namely  $S^\Lambda$ .  $\Lambda^c$  will denote the complementary of  $\Lambda$  in  $\Gamma$ , namely  $\Gamma \setminus \Lambda$ .

**Definition 7.1.** *A finite range, translation-invariant potential  $\{J_A\}_{A \subset \Gamma}$  is a family of real functions indexed by the non empty finite subsets of  $\Gamma$  verifying the following properties:*

1.  $J_A : \Omega_A \rightarrow \mathbb{R}$ .
2. For all  $A \subset \Gamma$  and all  $x \in \Gamma$ :

$$J_A(\sigma) = J_{A+x}(\eta) \quad \text{if} \quad \sigma(y+x) = \eta(y) \quad \forall y \in A.$$

3. There exists a positive  $r > 0$  such that  $J_A = 0$  if  $\text{diam } A > r$ , called range of interaction.

Given a finite range, translation-invariant potential, we can define a Hamiltonian for each finite lattice  $\Lambda \subset \Gamma$  and each boundary condition  $\tau \in \Omega_{\Lambda^c}$  by

$$H_\Lambda^\tau(\sigma) = - \sum_{A \cap \Lambda \neq \emptyset} J_A(\sigma \times \tau) \quad \forall \sigma \in \Omega_\Lambda,$$

where  $\sigma \times \tau$  is the configuration that agrees with  $\sigma$  over  $\Lambda$  and with  $\tau$  over  $\Lambda^c$ . For each such Hamiltonian, we define the Gibbs state state as

$$\mu_\Lambda^\tau(\sigma) = (Z_\Lambda^\tau)^{-1} \exp(-H_\Lambda^\tau(\sigma)),$$

where  $Z_\Lambda^\tau$  is a normalizing constant.<sup>9</sup> The convex hull of the set of Gibbs states over  $\Lambda$  will be denoted by  $\mathcal{G}(\Lambda)$ :

$$\mathcal{G}(\Lambda) = \text{conv}\{\mu_\Lambda^\tau \mid \tau \in \Omega_{\Lambda^c}\}.$$

**Definition 7.2.** The Glauber dynamics for a potential  $J$  is the Markov process on  $\Omega_\Lambda$  with the following generator:

$$(Q_\Lambda f)(\sigma) = \sum_{x \in \Lambda} c_J(x, \sigma) \nabla_x f(\sigma),$$

where  $\nabla_x f(\sigma)$  is defined as  $f(\sigma^x) - f(\sigma)$ , and  $\sigma^x$  is the configuration obtained by flipping the spin at position  $x$ :

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } x \neq y \\ -\sigma(x) & \text{if } x = y. \end{cases}$$

The numbers  $c_J(x, \sigma)$  are called transition rates and must verify the following assumptions:

1. *Positivity and boundedness:* There exist positive constants  $c_m$  and  $c_M$  such that:

$$0 < c_m \leq c_J(x, \sigma) \leq c_M < \infty \quad \forall x, \sigma.$$

2. *Finite range:*  $c_J(x, \cdot)$  depends only on spin values in  $b_r(x)$ .

3. *Translation invariance:* for all  $k \in \Gamma$ ,

$$c_J(x, \sigma') = c_J(x + k, \sigma) \quad \text{if } \sigma'(y) = \sigma(y + k) \quad \forall y.$$

4. *Detailed balance:* for all  $x \in \Gamma$  and all  $\sigma$

$$\exp\left(-\sum_{A \ni x} J_A(\sigma)\right) c_J(x, \sigma) = c_J(x, \sigma^x) \exp\left(-\sum_{A \ni x} J_A(\sigma^x)\right).$$

These assumptions are sufficient to ensure that  $Q_\Lambda$  generates a Markov process which has the Gibbs states over  $\Lambda$  as stationary points.

**Definition 7.3.** A quantum embedding of the classical Glauber dynamics for a potential  $J$  is generated by the following Lindblad operators

$$L_{x, \eta} = \sqrt{c_J(x, \eta)} |\eta^x\rangle \langle \eta| \otimes \mathbb{1}, \quad \forall x \in \Lambda, \forall \eta \in \Omega_{b_x(r)}; \quad (26)$$

$$\mathcal{L}_{x, \eta}(\rho) = L_{x, \eta} \rho L_{x, \eta}^* - \frac{1}{2} \{\rho, c_J(x, \eta) |\eta\rangle \langle \eta|\};$$

$$\mathcal{L}_\Lambda(\rho) = \sum_{x \in \Lambda} \sum_{\eta} L_{x, \eta} \rho L_{x, \eta}^* - \frac{1}{2} \{\rho, K\}, \quad K = \sum_{\sigma} \left( \sum_x c_J(x, \sigma) \right) |\sigma\rangle \langle \sigma|; \quad (27)$$

plus a dephasing channel acting independently and uniformly on all sites  $x \in \Lambda$ :

$$D_{x, 0} = \sqrt{\gamma} |0\rangle \langle 0|, \quad D_{x, 1} = \sqrt{\gamma} |1\rangle \langle 1|, \quad \mathcal{D}(\rho) = \sum_{x \in \Lambda} \sum_{i=0,1} D_{x, i} \rho D_{x, i}^* - |\Lambda| \gamma \rho. \quad (28)$$

<sup>9</sup>Following [37], in our notation we have incorporated the usual inverse temperature parameter  $\beta$  directly into the potential  $J$ .

$\mathcal{L}_\Lambda$  verifies translational invariance because the transition rates  $c_J$  do, so this family of Lindbladians is uniform.

*Observation 7.4.* Take  $|\alpha\rangle\langle\beta|$  an element of the computational basis, and call  $d(\alpha, \beta)$  the Hamming distance between  $\alpha$  and  $\beta$ . Then it holds that

$$\mathcal{D}(|\alpha\rangle\langle\beta|) = -\gamma d(\alpha, \beta) |\alpha\rangle\langle\beta|.$$

In other words,  $\mathcal{D}$  is a Schur multiplier in the computational basis, with Schur matrix given by  $(-\gamma d(\alpha, \beta))_{\alpha, \beta}$ .

On the other hand, we have that for all  $x$ :

$$\sum_{\eta \in \Omega_{b_x(r)}} \mathcal{L}_{x, \eta}(|\alpha\rangle\langle\beta|) = \begin{cases} c_J(x, \alpha) (|\alpha^x\rangle\langle\beta^x| - |\alpha\rangle\langle\beta|) & \text{if } \alpha|_{b_x(r)} = \beta|_{b_x(r)}, \\ -\frac{1}{2} (c_J(x, \alpha) + c_J(x, \beta)) |\alpha\rangle\langle\beta| & \text{otherwise.} \end{cases} \quad (29)$$

Since  $d(\alpha^x, \beta^x) = d(\alpha, \beta)$ ,  $[\mathcal{D}, \sum_{\eta} \mathcal{L}_{x, \eta}] = 0$  for all  $x \in \Lambda$ , and in particular  $\mathcal{D}$  and  $\mathcal{L}_\Lambda$  commute.

This quantum dissipative system inherits various properties from its classical counterpart.

**Definition 7.5.** Let  $\mu$  be a full-rank positive state. Call

$$\Gamma_\mu(\rho) = \mu^{\frac{1}{2}} \rho \mu^{\frac{1}{2}}.$$

We say that  $\mathcal{L}$  is in detailed balance [46] with respect to  $\mu$  if  $\Gamma_\mu \circ \mathcal{L} = \mathcal{L}^* \circ \Gamma_\mu$ .

**Proposition 7.6.** Let  $\mu_\Lambda^\tau$  be a Gibbs state over  $\Lambda$ . Then  $\mathcal{L}_\Lambda$  and  $\mathcal{D}$  are in detailed balance with respect to  $\mu_\Lambda^\tau$ .

*Proof.* Note that  $\Gamma_{\mu_\Lambda^\tau}$  is a Schur multiplier in the computational basis:

$$\Gamma_{\mu_\Lambda^\tau}(|\eta_1\rangle\langle\eta_2|) = \mu_\Lambda^\tau(\eta_1)^{\frac{1}{2}} \mu_\Lambda^\tau(\eta_2)^{\frac{1}{2}} |\eta_1\rangle\langle\eta_2|.$$

From the detailed balance condition for the transition rates  $c_J(x, \sigma)$ , it follows that for all  $x \in \Lambda$ , denoting  $\mathcal{L}_x = \sum_{\eta \in \Omega_{b_x(r)}} \mathcal{L}_{x, \eta}$ ,

$$\begin{aligned} \Gamma_{\mu_\Lambda^\tau} \circ \mathcal{L}_x \circ \Gamma_{\mu_\Lambda^\tau}^{-1}(|\eta_1\rangle\langle\eta_2|) &= \delta_{\eta_1, \eta_2}^x \left( c_J(x, \eta_1) \frac{\mu_\Lambda^\tau(\eta_1^x)}{\mu_\Lambda^\tau(\eta_1)} |\eta_1^x\rangle\langle\eta_2^x| - \frac{c_J(x, \eta_1) + c_J(x, \eta_2)}{2} |\eta_1\rangle\langle\eta_2| \right) \\ &= \delta_{\eta_1, \eta_2}^x c_J(x, \eta_1^x) |\eta_1^x\rangle\langle\eta_2^x| - \frac{c_J(x, \eta_1) + c_J(x, \eta_2)}{2} |\eta_1\rangle\langle\eta_2| \\ &= \mathcal{L}_x^*(|\eta_1\rangle\langle\eta_2|), \end{aligned}$$

where

$$\delta_{\eta_1, \eta_2}^x = \begin{cases} 1 & \text{if } \eta_1|_{b_x(r)} = \eta_2|_{b_x(r)} \\ 0 & \text{otherwise.} \end{cases}$$

To prove detailed balance for  $\mathcal{D}$ , note that Schur multipliers commute, thus  $[\mathcal{D}, \Gamma_\mu] = 0$ . This, together with the fact that  $\mathcal{D}^* = \mathcal{D}$ , implies that  $\mathcal{D}$  is in detailed balance w.r.t.  $\mu_\Lambda^\tau$ .  $\square$

The above proposition implies that Gibbs states are stationary states for the quantum Glauber dynamics. Let us prove that there are no other fixed points apart from the classical ones (i.e. states that are diagonal in the computational basis). Clearly,  $\mathcal{D}$  has all classical states as stationary points. We just have to check  $\mathcal{L}_\Lambda$ .

**Proposition 7.7.** *The set of fixed points of  $\mathcal{L}_\Lambda$  is equal to  $\mathcal{G}(\Lambda)$ , the set of Gibbs states over  $\Lambda$ .*

*Proof.* Let  $\rho$  be a fixed point of  $\mathcal{L}_\Lambda$ . We want to prove that  $\rho$  is diagonal, i.e. that it is of the form

$$\rho = \sum_{\sigma} p_{\sigma} |\sigma\rangle\langle\sigma|.$$

Consider a non-diagonal element  $|\alpha\rangle\langle\beta|$ , and suppose  $\alpha(x) \neq \beta(x)$  for some  $x \in \Lambda$ . Then, from equation (29), we have that for all  $y \in b_x(r)$ ,

$$\mathcal{L}_y(|\alpha\rangle\langle\beta|) = -\frac{1}{2}(c_J(y, \alpha) + c_J(y, \beta)) |\alpha\rangle\langle\beta|.$$

For  $y \notin b_x(r)$ ,  $\mathcal{L}_y$  is not supported on  $x$ , and thus cannot change the configuration there. This implies that the evolution cannot change the configurations over the set  $\Delta(r)$ , where  $\Delta = \{x \in \Lambda \mid \alpha(x) \neq \beta(x)\}$ . In turn, this implies that  $\mathcal{L}_\Delta$  commutes with  $\mathcal{L} - \mathcal{L}_\Delta$  (since it acts as a Schur multiplier whose entries depend only on the sites in  $\Delta(r)$ ). Finally, this means that

$$\begin{aligned} \|e^{t\mathcal{L}_\Lambda}(|\alpha\rangle\langle\beta|)\|_1 &\leq \|e^{t\mathcal{L}_\Delta}(|\alpha\rangle\langle\beta|)\|_1 = \exp\left(-t\frac{1}{2}\left(\sum_{x \in \Delta} c_J(x, \alpha) + c_J(x, \beta)\right)\right) \\ &\leq \exp\left(-t\frac{1}{2}c_m d(\alpha, \beta)\right) \rightarrow 0. \end{aligned}$$

Since the off-diagonal elements are killed,  $\rho$  must be of the form  $\sum_{\sigma} p_{\sigma} |\sigma\rangle\langle\sigma|$ . Writing the equation  $\mathcal{L}_\Lambda(\rho) = 0$  we obtain

$$\sum_{\sigma} \sum_x c_J(x, \sigma) p_{\sigma} |\sigma^x\rangle\langle\sigma^x| - \sum_{\sigma} \sum_x c_J(x, \sigma) p_{\sigma} |\sigma\rangle\langle\sigma| = 0,$$

which implies

$$\sum_x c_J(x, \sigma^x) p_{\sigma^x} = \sum_x p_{\sigma} c_J(x, \sigma).$$

The last equation is simply a rewriting of the fact that  $(p_{\sigma})$  is a stationary distribution for  $Q_\Lambda$ , that is, it is exactly a Gibbs state on  $\Lambda$ .  $\square$

Since  $\mathcal{L}_\Lambda$  and  $\mathcal{L}_\Lambda + \mathcal{D}$  have the same stationary distributions, even locally, all properties that depend just on the structure of the fixed points sets will be shared by both: this is the case, for example, of frustration freeness and LTQO, which we will prove next.

**Proposition 7.8.**  *$\mathcal{L}_\Lambda$  (and consequently  $\mathcal{L}_\Lambda + \mathcal{D}$ ) is frustration free.*

*Proof.* By the previous proposition, we have that  $\mathcal{X}_{\mathcal{L}_\Lambda} = \mathcal{G}(\Lambda)$ . We know [33] that for Gibbs states it holds that

$$\Delta \subset \Lambda \Rightarrow \mathcal{G}(\Lambda) \subset \mathcal{G}(\Delta),$$

but this is exactly the frustration-freeness condition for  $\mathcal{L}_\Lambda$ .  $\square$

Let us recall an important definition from [37].

**Definition 7.9.** We say that the Gibbs measures in  $\mathcal{G}(\Lambda)$  satisfy the weak mixing condition in  $V \subset \Lambda$  if there exist constants  $C$  and  $m$  such that, for every subset  $\Delta \subset V$ , the following holds:

$$\sup_{\tau, \tau' \in \Omega_{V^c}} \left\| \mu_{V, \Delta}^\tau - \mu_{V, \Delta}^{\tau'} \right\|_1 \leq C \sum_{\substack{x \in \Delta, \\ y \in \partial_r^+ V}} e^{-m \text{dist}(x, y)}, \quad (30)$$

where  $\partial_r^+ V = \{x \in V^c \mid \text{dist}(x, V) \leq r\}$  and  $\mu_{V, \Delta}^\tau = \text{tr}_{V \setminus \Delta} \mu_V^\tau$ .

**Proposition 7.10.** If  $\mathcal{G}(\Lambda)$  verifies the weak mixing condition for each  $V \subset \Lambda$ , then  $\mathcal{L}_\Lambda$  (and consequently  $\mathcal{L}_\Lambda + \mathcal{D}$ ) verifies LTQO.

*Proof.* Take  $A \subset \Lambda$ ,  $\ell \geq 0$ , and call  $V = A(\ell)$ . The weak mixing condition for  $V$  implies that there exist constants  $C$  and  $m$  such that

$$\sup_{\tau, \tau' \in \Omega_{V^c}} \left\| \mu_{V, A}^\tau - \mu_{V, A}^{\tau'} \right\|_1 \leq C \sum_{\substack{x \in A, \\ y \in \partial_r^+ V}} e^{-m \text{dist}(x, y)} \leq C e^{-m\ell} |A| |\partial_r^+ A(\ell)|.$$

This is the LTQO condition with  $\Delta_0(\ell) = C e^{-m\ell} |A| |\partial_r^+ A(\ell)|$ . The bound, proven for states of the form  $\mu_V^\tau$ , can be extended by convexity to all  $\mathcal{G}(V)$ . Let  $\eta_0, \eta_1 \in \mathcal{G}(V)$ . By definition,  $\eta_0$  and  $\eta_1$  are convex combination of states of the form  $\mu_V^\tau$ , thus we can write

$$\eta_0 = \sum_i p_i \mu_V^{\tau_i}, \quad \eta_1 = \sum_j q_j \mu_V^{\sigma_j}, \quad \sum_i p_i = \sum_j q_j = 1; \quad p_i, q_j \geq 0.$$

Then we have

$$\begin{aligned} \|\eta_{0,A} - \eta_{1,A}\|_1 &= \left\| \sum_i p_i \mu_{V,A}^{\tau_i} - \sum_j q_j \mu_{V,A}^{\sigma_j} \right\|_1 \\ &= \left\| \sum_i p_i \left( \sum_j q_j \mu_{V,A}^{\tau_i} \right) - \sum_j q_j \left( \sum_i p_i \mu_{V,A}^{\sigma_j} \right) \right\|_1 \leq \sum_{i,j} p_i q_j \left\| \mu_{V,A}^{\tau_i} - \mu_{V,A}^{\sigma_j} \right\|_1 \\ &\leq \sup_{\tau, \sigma} \left\| \mu_{V,A}^\tau - \mu_{V,A}^\sigma \right\|_1. \end{aligned}$$

□

Finally, we want to show that the contraction of the semigroup generated by  $\mathcal{L}_\Lambda + \mathcal{D}$  can be controlled by the contraction of the classical Glauber dynamics. To fix notation, call  $\mathcal{C} : \mathcal{A}_\Lambda \rightarrow \mathcal{A}_\Lambda$  the projector on the computational basis diagonal.  $\mathcal{C}$  is a completely positive, trace preserving map, and it also verifies  $\mathcal{C} = \lim_{t \rightarrow \infty} \exp(t\mathcal{D})$ . Since  $\mathcal{L}_\Lambda$  commutes with  $\mathcal{D}$ , it also commutes with  $\mathcal{C}$ . Then we can prove the following

**Lemma 7.11.** If  $T_t = \exp(t(\mathcal{L}_\Lambda + \mathcal{D}))$ , then

$$\eta(T_t) \leq \eta(T_t \circ \mathcal{C}) + \eta(\exp(t\mathcal{D})). \quad (31)$$

*Proof.* Fix an initial state  $\rho$ . Then we can write

$$\begin{aligned}\|T_t(\rho) - T_\infty(\rho)\|_1 &\leq \|T_t \circ \mathcal{C}(\rho) - T_\infty(\rho)\|_1 + \|T_t \circ (1 - \mathcal{C})(\rho)\|_1 \\ &\leq \|T_t \circ \mathcal{C}(\rho) - T_\infty \circ \mathcal{C}(\rho)\|_1 + \|\exp(t\mathcal{D}) \circ (1 - \mathcal{C})(\rho)\|_1 \\ &\leq \eta(T_t \circ \mathcal{C}) + \eta(\exp(t\mathcal{D})),\end{aligned}$$

where we have used the fact that  $\mathcal{L}_\Lambda$  and  $\mathcal{D}$  commute, and that the fixed points of  $\mathcal{L}_\Lambda$  are invariant under  $\mathcal{C}$ .  $\square$

We know, because of theorem 6.1, that

$$\eta(\exp(t\mathcal{D})) \leq |\Lambda| e^{-\frac{\gamma}{2}t}, \quad (32)$$

and this implies the following result.

**Corollary 7.12.** *If the classical Glauber dynamics verifies global rapid mixing, then also the quantum embedded Glauber dynamics generated by  $\mathcal{L}_\Lambda + \mathcal{D}$  does.*

*Observation 7.13.* Convergence rates of classical Glauber dynamics are a well studied subject. It is known that, in some regimes, classical Glauber dynamics verifies a Log Sobolev inequality with system-size independent Log Sobolev constant (for a review on the subject see [37]). In such situations the classical chain has a logarithmic mixing time, and thus verifies global rapid mixing.

For this class of classical dynamical systems it is possible to apply our main result 4.6. In particular, we can arbitrary perturb the transition rates  $c_J(x, \sigma)$  by some  $e(x, \sigma)$ , not necessary preserving detailed balance. If we call  $\mathcal{E}$  the maximum of  $|e(x, \sigma)|$ , the difference between the perturbed and the original evolution of local observables can be bounded by  $\mathcal{E}$  times a factor depending on the size of the support of the observables taken into account.

**Theorem 7.14.** *Let  $Q_\Lambda$  the generator of a classical Glauber dynamics, having a unique fix point and verifying a Log Sobolev inequality with constant independent of system size. Let  $E$  be the generator of another classical Markov process of the form*

$$(Ef)(\sigma) = \sum_{x \in \Lambda} e(x, \sigma) \nabla_x f(\sigma).$$

*Suppose that  $\mathcal{E} = \sup_{x, \sigma} |e(x, \sigma)| < \infty$  and that  $e(x, \cdot)$  has bounded support uniformly in  $x$ . Denote by  $T_t$  the evolution generated by  $Q_\Lambda$  and by  $S_t$  the evolution generated by  $Q_\Lambda + E$ . Then, for each function  $f$  supported on  $A \subset \Lambda$ , it holds that*

$$\|T_t(f) - S_t(f)\|_\infty \leq c(|A|) \|f\|_\infty \mathcal{E},$$

*for some  $c(\cdot)$  independent of system size and polynomially growing.*

*Observation 7.15.* It is known [35, 36] that the Ising model on  $\mathbb{Z}^2$  or  $(\mathbb{Z}/n\mathbb{Z})^2$  has a system size independent Log Sobolev constant for high temperatures (when the inverse temperature  $\beta$  is lower than the critical value  $\beta_c$ ), or at any temperature in presence of an external magnetic field. In this regime the Glauber dynamics sampling the Ising model is stable (in the sense of theorem 4.6).

## 8 Conclusions and open questions

We have shown stability of the evolution of local observables under local perturbations for frustration-free Lindblad evolutions with unique fixed points, under the assumption that the mixing time of the system scales logarithmically with the system size. In the general case of non-uniform Lindbladians, in which no translational invariance is required, additional assumptions of topological order and truncated rapid mixing are necessary. We have shown the equivalence between LTQO and the uniqueness of the fixed point for such fast-converging systems. It should be emphasized that Log Sobolev inequalities provide the exact type of convergence time estimates needed for stability.

The most important open question is what happens outside the rapid mixing regime, for example for systems for which the local convergence time depends on the global system size. This seems to be the case of state engineering of degenerate topological ordered states, such as topologically protected quantum codes.

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## Appendix A The non-stable example

The following example will verify all hypotheses of theorem 4.6, except for truncated rapid mixing and belonging to a uniform family, and will be shown to be unstable. Interestingly, the system *is* globally rapid mixing. This example is similar to the globally gapped but not locally gapped example in [39]. We will show that the characteristics of the dynamics are essentially determined by a classical Markov chain embedded into the Lindbladian. For a general review on convergence of Markov chains, see [32].

*Example A.1.* Consider a chain of  $2N$  qubits, and define a Hamiltonian  $H^{2N} = \sum_{k=1}^{2N} H_{k,k+1}$ , where  $H_{k,k+1}$  acts only on the qubits  $(k, k+1)$  and is defined as follows:

$$H_{k,k+1} = \frac{2}{3N} U_{k,k+1} + |0\rangle\langle 0|_k \otimes |0\rangle\langle 0|_{k+1} + |1\rangle\langle 1|_k \otimes |1\rangle\langle 1|_{k+1},$$

where

$$U_{k,k+1} = \begin{cases} |1\rangle\langle 1|_k \otimes |0\rangle\langle 0|_{k+1} & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

Direct verification shows that  $H^{2N}$  verifies the following properties:

1.  $H^{2N}$  is local, frustration-free, gapped and verifies LTQO.
2.  $|0101\dots 01\rangle$  is the unique groundstate.



3.  $|1010 \dots 10\rangle$  is the unique lowest-energy excited state, with energy  $2/3$ .
4. Any other state has energy bigger than 1.

It is easy to see that this Hamiltonian is unstable under small local perturbation, because we can perturb it with  $U = -\sum_k \frac{2}{3N} U_{k,k+1}$  and the resulting Hamiltonian  $H^{2N} + U$  has degenerate groundstate space (we have lowered the energy of the first excitation to 0).

A Lindbladian constructing the groundstate of  $H^{2N}$  is the following:

$$\mathcal{L}^{2N} = \sum_{k=1}^{2N} \mathcal{L}_{k,1} + \mathcal{L}_{k,2} + \frac{2}{3N} \mathcal{L}_{k,3},$$

where  $\mathcal{L}_{k,i}$  are local terms generated by the following corresponding Lindblad operators: if  $k$  is odd, then

$$\begin{aligned} L_{k,1} &= \sigma_x^{k+1} |0\rangle\langle 0|_k \otimes |0\rangle\langle 0|_{k+1}, \\ L_{k,2} &= \sigma_x^k |1\rangle\langle 1|_k \otimes |1\rangle\langle 1|_{k+1}, \\ L_{k,3} &= \sigma_x^k \otimes \sigma_x^{k+1} |1\rangle\langle 1|_k \otimes |0\rangle\langle 0|_{k+1}; \end{aligned}$$

if  $k$  is even, then

$$\begin{aligned} L_{k,1} &= \sigma_x^k |0\rangle\langle 0|_k \otimes |0\rangle\langle 0|_{k+1}, \\ L_{k,2} &= \sigma_x^{k+1} |1\rangle\langle 1|_k \otimes |1\rangle\langle 1|_{k+1}, \\ L_{k,3} &= 0. \end{aligned}$$

We want to show that  $\mathcal{L}^{2N}$  has spectral gap of  $1/3$ . Nonetheless, it fails to verify truncated rapid mixing: if we consider truncations of  $\mathcal{L}^{2N}$  to subsystems of  $k$  sites, the corresponding spectral gap is of order  $O(k/N)$ .

Consider the elements of the computational basis, which are  $\{|\alpha\rangle\langle\beta|\}$  for each  $\alpha, \beta \in \{0, 1\}^{2N}$ . By direct calculation, we can see that the *diagonal elements*  $\{|\alpha\rangle\langle\alpha|\}$  span a subspace invariant under  $\mathcal{L}^{2N}$ . This is the dynamics of a classical Markov process that is embedded into our quantum dissipative process.

Let us analyze the evolution of this invariant subspace first. If we group the sites in pairs, forming a chain of  $N$  4-dimensional systems, starting with the pair  $(1, 2)$ , we see that each of the  $\mathcal{L}_{k,i}$  can change only the value of one of the pairs. This is trivial for  $k$  odd; when  $k$  is even,  $\mathcal{L}_{k,i}$  is supported on two different pairs, but acts as a controlled-amplitude damping channel on only one of the two pairs. We can then easily write down the transition matrix for the diagonal states:

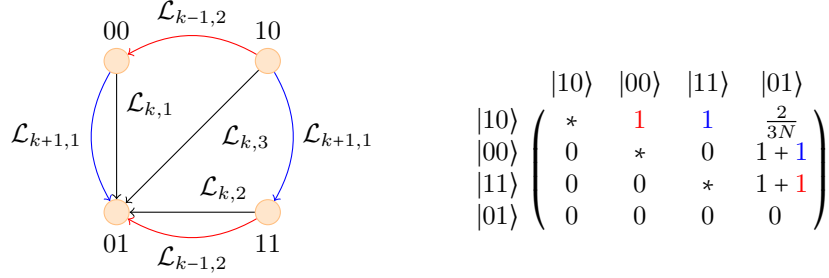


Figure 1: The section of transition matrix for  $\mathcal{L}^{2N}$  on the pair  $(k, k+1)$  with  $k$  odd. The blue and the red transitions are present depending on the nearby sites: the blue ones if there is a 0 on the right, the red ones if there is a 1 on the left. Asterisks in the diagonal are such that the sum of each row is zero.

Figure 1 shows a local section of the transition graph and its corresponding infinitesimal generator matrix. The global graph is given by the cartesian product of these sections, and since they are all acyclic (as directed graphs), the global graph is acyclic too. To see this, let us define the following

$$\begin{aligned}
 Q &= \begin{matrix} & |10\rangle & |00\rangle & |11\rangle & |01\rangle \\ \begin{matrix} |10\rangle \\ |00\rangle \\ |11\rangle \\ |01\rangle \end{matrix} & \begin{pmatrix} -\frac{2}{3N} & 0 & 0 & \frac{2}{3N} \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}, & Q_b = \begin{matrix} & |10\rangle & |00\rangle & |11\rangle & |01\rangle \\ \begin{matrix} |10\rangle \\ |00\rangle \\ |11\rangle \\ |01\rangle \end{matrix} & \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}, \\
 Q_r &= \begin{matrix} & |10\rangle & |00\rangle & |11\rangle & |01\rangle \\ \begin{matrix} |10\rangle \\ |00\rangle \\ |11\rangle \\ |01\rangle \end{matrix} & \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}, & S_b = |0\rangle\langle 0| \otimes \mathbb{1}, \quad S_r = \mathbb{1} \otimes |1\rangle\langle 1|.
 \end{aligned}$$

The transition matrix for the diagonal states is then given by

$$\begin{aligned}
 Q^{2N} &= \sum_{k=1}^N \mathbb{1}_{1\dots 2k-1} \otimes Q \otimes \mathbb{1}_{2k+1,\dots,2N} \\
 &\quad + \sum_{k=1}^N \mathbb{1}_{1\dots 2k-1} \otimes Q_b \otimes S_b \otimes \mathbb{1}_{2k+3,\dots,2N} \\
 &\quad + \sum_{k=1}^N \mathbb{1}_{1\dots 2k-1} \otimes S_r \otimes Q_r \otimes \mathbb{1}_{2k+3,\dots,2N}.
 \end{aligned}$$

Therefore  $Q^{2N}$  is upper triangular, the elements on the diagonal are the eigenvalues, and consequently are also the eigenvalues of  $\mathcal{L}^{2N}$  restricted to the diagonal states. We see that the smallest non-zero eigenvalue, corresponding to the state  $|1010\dots 10\rangle$ , is  $\frac{2}{3}$ . Furthermore, it is easy to see that the diameter of the graph of the transitions of  $Q^{2N}$  is  $N$ , and this implies that the mixing time for  $\mathcal{L}^{2N}$  is of order  $O(\log N)$ <sup>10</sup>.

<sup>10</sup>This can be seen from the upper triangular form of  $Q^{2N}$ , noticing that the polynomials

Let us now concentrate on off-diagonal elements of the computational basis, i.e. matrices of the form  $|\alpha\rangle\langle\beta|$  where  $\alpha \neq \beta \in \{0,1\}^{2N}$ . Fix one, and call  $\Delta = \{k \mid \alpha(k) \neq \beta(k)\}$ . Then, by direct calculation, we have that

$$\mathcal{L}_k(|\alpha\rangle\langle\beta|) = -\frac{1}{2}\{\rho, H_{k,k+1}\} \quad \text{if } k \text{ or } k+1 \in \Delta.$$

Given  $k \in \Delta$ , the only operators that have range over  $k$  are  $\mathcal{L}_k$  and  $\mathcal{L}_{k-1}$ , but since they only act as Schur multipliers on  $|\alpha\rangle\langle\beta|$ , the evolution cannot change the value of  $\alpha$  and  $\beta$  over  $\Delta$ . This implies that the terms of  $\mathcal{L}^{2N}$  whose support intersects  $\Delta$  are simply contracting the evolving matrix, while the ones supported on  $\Delta^c$  are acting as the classical Markov chain generated by  $Q^{2N}$  (since, by definition of  $\Delta^c$ , on such sites  $|\alpha\rangle\langle\beta|$  is locally a diagonal element of the computational basis). Matrices  $|\alpha\rangle\langle\beta|$  who share the same set  $\Delta$  span an invariant subspace for  $\mathcal{L}^{2N}$ .

By permuting the basis in all such invariant subspaces, we can write  $\mathcal{L}^{2N}$  fully in an upper-triangular form, similarly to what we did before on the “globally diagonal” subspace. The eigenvalues of  $\mathcal{L}^{2N}$  are then simply given by the numbers

$$\lambda_{\alpha,\beta} = -\frac{1}{2}\{|\alpha\rangle\langle\beta|, H^{2N}\}, \quad \forall \alpha, \beta \in \{0,1\}^{2N},$$

and consequently the spectrum of  $\mathcal{L}^{2N}$  and the spectrum of  $H^{2N}$  are related as follows (where the sum here is intended as a sum of sets)

$$\sigma(\mathcal{L}^{2N}) = \frac{\sigma(H^{2N}) + \sigma(H^{2N})}{2}.$$

Since the spectral gap of  $H^{2N}$  is  $2/3$ , the smallest non-zero eigenvalue for  $\mathcal{L}$  is  $1/3$ .

The previous analysis also shows that the fixed points of  $\mathcal{L}^{2N}$  are exactly the stationary states of  $Q^{2N}$ , and thus  $\mathcal{L}^{2N}$  has  $|01\dots 01\rangle\langle 01\dots 01|$  as unique fixed point. It is then easy to check that  $\mathcal{L}^{2N}$  verifies LTQO and frustration-freeness.

We have therefore shown that  $\mathcal{L}^{2N}$  verifies frustration-freeness, LTQO, and is globally gapped. By adding a uniform dephasing noise on all sites  $\mathcal{D}$ , by the same argument used in lemma 7.11, one can prove that  $\mathcal{L}^{2N} + \mathcal{D}$  has the same fixed points as  $\mathcal{L}^{2N}$ , and its convergence rate is given by the classical Markov chain  $Q^{2N}$  embedded in the diagonal of  $\mathcal{L}^{2N}$ . Since we showed that the global mixing time of  $Q^{2N}$  scales as  $O(\log N)$ , we can conclude that  $\mathcal{L}^{2N} + \mathcal{D}$  verifies global rapid mixing.

Nonetheless it fails to verify truncated rapid mixing, since the local interaction terms of  $H^{2N}$  are not gapped, and by a similar argument one can prove that

$$\sigma(\mathcal{L}_{(0\dots,2k)}^{2N}) = \frac{\sigma(H_{(0\dots,2k)}^{2N}) + \sigma(H_{(0\dots,2k)}^{2N})}{2}.$$

Consequently, the spectral gap of  $\mathcal{L}_{(0\dots,2k)}^{2N}$  is  $k/3N$ .

$\mathcal{L}^{2N} + \mathcal{D}$  is clearly an unstable system, since by removing the terms  $\mathcal{L}_{k,3}$  (which have strength  $O(1/N)$ ), a new fixed point appears,  $|10\dots 10\rangle\langle 10\dots 10|$ , locally orthogonal to the original one. Similarly to example 4.12, the perturbed system does verify truncated rapid mixing, but fails to verify LTQO, and is obviously unstable.

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appearing in  $e^{tQ^{2N}}$  have degree of at most the diameter of the transition graph.

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